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# SLOW AND FAST MOTION OF CRACKS IN INELASTIC SOLIDS

by

Michael P. Wnuk and George C. Sih

**PART I: Slow Growth of Cracks in a Rate Sensitive Tresca Solid.**

**PART II: Dynamic Crack Represented by the Dugdale Model.**

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Part II by G.C. Sih: Dynamic Crack Represented by the Dugdale Model.

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Part I

SLOW GROWTH OF CRACKS IN A RATE SENSITIVE TRESCA SOLID

by

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(Abstract)

The paper proposes an extension of the classical theory of fracture to viscoelastic and elastic-plastic materials in which the plasticity effects are confined to a narrow band encompassing the crack front.

It is suggested that the Griffith-Irwin criterion of fracture, which requires that the energy release rate computed for a given boundary value problem equals the critical threshold, ought to be replaced by a differential equation governing the slow growth of a crack prior to the onset of rapid propagation. A new term which enters the equation of motion in the dissipative media is proportional to the energy lost within the end sections of the crack, and thus it reflects the extent of inelastic behavior of a solid.

A concept of "apparent" surface energy is introduced to account for the geometry dependent and the rate dependent phenomena which influence toughness of an inelastic solid.

Three hypotheses regarding the condition for fracture in the subcritical range of load are compared. These are: (a) constant fracture energy (Cherepanov), (b) constant opening displacement at instability (Morozov) and (c) final stretch criterion (Wnuk).

# Slow Growth of Cracks in a Rate Sensitive Tresca Solid,

by

Michael P. Wnuk

Theoretical analysis shows that the amount of slow growth occurring in a plane stress tensile specimen, subjected to a subcritical stress intensity level, is affected by the following parameters:

- (1) ductility and rheological sensitivity of the material,
- (2) rate of loading,
- (3) initial crack size,
- (4) geometrical configuration of the test.

Some of these factors were included in the earlier semi-empirical approaches of Krafft et al. [3], Brown and Srawley [1], and in the only available exact treatment of slow growth under tearing mode of deformation due to McClintock [4], McClintock and Irwin [5] and Rice [6]. Here Rice's idea of a universal R-curve is re-examined. It turns out that the curve is universal, that is independent of geometry and the initial crack size, only in the limiting case of a rate-insensitive elastic-plastic solid. When time effects are accounted for in the constitutive equations of the matrix which contains the crack, the "universality" no longer holds.

The governing equation which describes the quasi-static extension of the crack under confined yielding condition is an integro-differential equation derived in [8]:

$$\begin{aligned}
 (1) \quad R - \sqrt{R(R - \Delta)} + \Delta \frac{dR}{d\ell} + \frac{\Delta}{2} \ln \left( \frac{\sqrt{R} + \sqrt{R - \Delta}}{\sqrt{R} - \sqrt{R - \Delta}} \right) + \\
 + \frac{\delta \Psi}{\Delta} R^2 \int_0^{\ell} \left\{ \sqrt{\rho(s)(\rho(s) - s)} - \frac{s}{2} \ln \left( \frac{\sqrt{\rho(s)} + \sqrt{\rho(s) - s}}{\sqrt{\rho(s)} - \sqrt{\rho(s) - s}} \right) \right\} ds = R_0
 \end{aligned}$$

Here  $R$  denotes the length of plastic zone ahead of the crack front,  $R_0$  is its value at the growth initiation,  $\Delta$  is the dimension of the process zone,  $\Psi$  is the normalized creep compliance. Both  $R$  and  $\Psi$  are functions of time  $t$ , or equivalently, functions of the current crack length  $\ell$ , which is treated as a time-like parameter. Other relations are

$$\epsilon = \Delta/R(t - \delta t) \text{ , } \delta t = \Delta/\dot{\ell} \text{ , } \dot{\ell} = d\ell/dt$$

$$\rho(s) = 1 + (\epsilon - s) dR/d\ell \text{ , } \delta\Psi = \Psi(\delta t) - \Psi(0)$$

The criterion for crack opening, which was employed in order to derive eq. (1), is that of "final stretch", cf. [8]. In contrast to the COD criterion the final stretch condition is path-dependent and thus it appears to withstand Rice's criticism [7] of earlier work on this subject by Cherepanov [2]. Postulating this criterion in [8] we required that the amount of stretching which occurs within the process zone during the time interval just prior to fracture is a material constant. Such an approach assumes nothing about the current tip displacement and the length of the associated plastic zone. In fact these two entities turn out to be functions of time and the loading history.

It should be noted that the final stretch criterion coincides with McClintock's criterion of critical strain attained over the Neuber domain. One may also add <sup>that</sup> in the limit case of steady-state propagation, both the COD and the final stretch criteria converge. The essential difference between them becomes obvious, though, within the subcritical range of applied stress intensity.

To make the problem mathematically tractable we assume further that the length of the process zone is small vs. the plastic zone size, i.e.

$\Delta/R \ll 1$ . Then eq. (1) reduces to

$$(2) \quad \frac{\Delta}{2} \cdot \ln\left(\frac{4R}{\Delta}\right) + \Delta \cdot [1 + CR(\partial R/\partial Q)^{-1}] \frac{dR}{d\ell} - \Delta C(\partial R/\partial Q)^{-1} R \frac{\partial R}{\partial \ell} = R_0$$

Here  $(dR/d\ell - \partial R/\partial \ell) \Delta (\partial R/\partial Q)^{-1}$  has been substituted for  $\delta\psi (= B\Delta/\dot{\ell})$ ; B denotes the slope of creep compliance at time zero, Q is the loading parameter and  $C = B/\dot{Q}$ . To illustrate applications of eq. (2) we shall integrate it numerically for the case of a crack contained in an infinite plate under tension  $\sigma$ ; then  $R = (1/2)Q^2(\ell)\ell$ ,  $Q = \pi\sigma/2Y$ . Since the ratios  $R/\Delta$  and  $\ell/\Delta$  are very large numbers, it is convenient to introduce the logarithmic function and to cast the eq. (2) into the following form

$$(3) \quad \frac{dY}{dX} = \frac{A - (1/2)Y + \frac{C}{\sqrt{2}} \exp(3/2)(Y - X)}{1 + \frac{C}{\sqrt{2}} \exp(1/2)(Y - X)} \exp(X - Y)$$

where

$$(4) \quad \begin{aligned} X &= \log(\ell/\Delta), & A &= (R_0/\Delta) - \log 2 \\ Y &= \log(R/\Delta), & Y &= Y(X) \end{aligned}$$

Equation (3) has been integrated numerically for a certain set of initial conditions, and the results are shown in Fig. 1. The figure illustrates the effect of rate sensitivity of the material and the rate of loading on the shape of the R-curve. It is seen that not only the slope of the curve is affected, but also pronounced changes in location of the ultimate instability point are observed. Examples of integration of the equation of motion when the visco-elastic dissipation is dominant, are shown in Fig. 2 and Fig. 3. Such a case of a "creeping crack" is considered in more detail in [9].

For engineering applications it is convenient to re-write the governing equation (2) in terms of the ratios  $R/R_\infty$  and  $\ell/R_\infty$ , where  $R_\infty$  denotes the steady-state limit of the plastic zone size, say  $R_\infty = \pi K_\infty^2/8Y^2$ . The symbol

$K_{\infty}$  denotes the maximum plane stress fracture toughness which would be attained in an ideal case, when the conditions of the test are such that the prior to failure slow growth is fully developed. Of course, the actual fracture toughness  $K_f$ , i.e. the value of  $K$  at which the rapid motion begins, is bracketed by the initiation toughness  $K_0$  and the maximum steady-state toughness  $K_{\infty}$ .

Normalizing the plastic zone size and the crack length as follows

$$(5) \quad R/R_{\infty} = \rho, \quad \ell/R_{\infty} = \zeta, \quad R_{\infty} = \pi K_{\infty}^2 / 8Y^2$$

changes eq. (2) into

$$(6a) \quad \frac{d\rho}{d\zeta} = \frac{(1/2)\ln(1/\rho) + C\rho^2/\zeta\sqrt{2\zeta\rho}}{1 + C\rho/\sqrt{2\zeta\rho}}, \quad \rho = (1/2)\zeta^2$$

or

$$(6b) \quad \frac{dQ}{d\zeta} = \frac{\ln(2/Q^2\zeta) - Q^2}{2\zeta Q(1 + CQ/2)}$$

for a crack in an infinite plate, and

$$(7a) \quad \frac{d\rho}{d\zeta} = \frac{(1/2)\ln(1/\rho) + C\rho^2\{1 + (p\zeta/\zeta_0)\tan(p\zeta/\zeta_0)\}/\zeta\sqrt{2\rho\zeta\sec(p\zeta/\zeta_0)}}{1 + C\rho/\sqrt{2\rho\zeta\sec(p\zeta/\zeta_0)}}$$

or

$$\rho = (1/2)\zeta^2 \zeta\sec(p\zeta/\zeta_0)$$

$$(7b) \quad \frac{dQ}{d\zeta} = \frac{\ln[2/Q^2\zeta\sec(p\zeta/\zeta_0)] - Q^2[1 + (p\zeta/\zeta_0)\tan(p\zeta/\zeta_0)]\sec(p\zeta/\zeta_0)}{2Q\zeta(1 + CQ/2)\sec(p\zeta/\zeta_0)}$$

for a crack traversing a panel of width  $2b$ . The initial crack length is given by  $\zeta_0 = \ell_0/R_{\infty}$  and  $p$  denotes the initial crack length to panel width ratio,  $p = \pi\ell_0/2b$ . The locus of terminal instability follows readily from eqs.

(6b) and (7b) if  $dQ/d\ell$  is set equal to zero. Then for a crack contained in an infinite plate one has

$$(8) \quad \zeta_f = 2/Q_f^2 \cdot \exp(Q_f^2)$$

while for a central crack in a finite width panel it is

$$(9) \quad \ln [2/Q_f^2 \zeta_f \sec(p\zeta_f/\zeta_0)] = Q_f^2 [1 + (p\zeta_f/\zeta_0) \tan(p\zeta_f/\zeta_0)] \sec(p\zeta_f/\zeta_0)$$

Interestingly the rate sensitivity  $C$  does not enter explicitly in the above relations. It is present here, though, in an implicit way, since both the critical load  $Q_f$  and the critical crack size  $\zeta_f$  are pronouncely affected by the rate sensitivity. This can be seen only after the integration of equations (6) and (7) is completed, see Fig. 4.

The effect of finite width on the amount of slow growth which takes place prior to failure is illustrated in Fig. 4, where  $Q$  vs.  $\zeta$  curves are shown for both infinite and finite plates at certain levels of loading rate  $\dot{Q}$ . Two different trends are observed: (1) the slow growth is enhanced when the panel width stays constant while the initial crack size is increased, and (2) the slow growth is diminished when initial crack size is kept constant but the panel width increases, see Figs. 5 and 6.

In general the amount of slow growth before the final instability sets in turns out to be a function of (1) ductility, (2) rate-sensitivity, (3) rate of loading (4) initial crack size, and (5) geometry of the test. Although no closed form solutions are available at this time, the influence of the foregoing factors has been investigated numerically, and the results are gathered in Figs. 5 through 8. The graphs were derived from a number of integrations of the governing eq. (2) performed on an IBM 360.

It should be noted that for the case of fast loading, or equivalently, for a rate insensitive solid ( $C \rightarrow 0$ ), the equation (2) which describes the R-curve degenerates to

$$(10) \quad \frac{d\rho}{d\zeta} = (1/2) \ln(1/\rho)$$



This has a closed form solution

$$(11) \quad \zeta - \zeta_0 = 2\{ei[\ln \rho_0] - ei[\ln \rho]\}$$

$$ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt$$

It is seen that the last equation supplies a universal relationship, since neither geometry nor the initial crack size are represented (the initial crack length enters only in form of the difference  $\zeta - \zeta_0$ ). Thus the shape of the resulting R vs.  $\ell$  curve will remain unaltered by these factors. On the other hand, the location of the terminal instability point will depend on a specific geometry of the test. It is so, because the instability state is determined by the point of tangential contact between the R-curve and a member of the family of curves originating at  $(R=0, \ell=0)$  and representing the variation of plastic zone size with crack length at certain fixed values of loading parameter. Slope and the shape of these curves will, of course, depend on the geometry of the problem, and so will the position of the instability point.

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## APPENDIX

### A. Fracture Criterion and Slow Growth of a Crack in an Elastic-plastic Solid.

A local criterion of fracture is postulated by proposing that the sum of work done at a fixed material point, which undergoes a deformation process while it traverses the Neuber section of the plastic zone, is a material property. Thus the crack will move onward if

$$(A1) \quad \int_{t-\delta t}^t S[x_p, \tau] \dot{u}[x_p, \tau] d\tau = \phi_*$$

Here,  $S[x_p, \tau]$  is the restraining stress at the control point P and time  $\tau$ , while  $\dot{u}[x_p, \tau]$  denotes the time-rate of the displacement at the point P and time  $\tau$ , and perpendicular to the crack plane. Time increment  $\delta t$  equals the time used by the crack front to pass through the Neuber zone (of characteristic length  $\Delta$ ) and thus it is related to the rate of crack growth as follows

$$(A2) \quad \delta t = \Delta / \dot{x}$$

Such a relation was used earlier by Glennie and Willis [A1] to describe a piece-wise linear approximation of an accelerating crack. The material property  $\phi_*$  has a dimension of energy and it can be related to the threshold fracture energy, i.e. the specific fracture energy at the onset of crack growth.

We assume further that the restraining stress is constant ( $=\gamma$ ) over the Neuber domain and that the work  $\phi_*$  can be expressed as a product of  $\gamma$  and the initiation displacement  $u_0$ , i.e. the tip displacement at which motion of the crack sets in. Under these assumptions criterion (A1) reduces to the "final stretch" condition which reads

$$(A3) \quad u(x_p, t) - u(x_p, t - \delta t) = u_0, \quad \delta t = \Delta/\dot{u}$$

In other words the increment of deformation generated at the point P, just before an infinitesimal element located at this point collapses, should remain constant during the slow propagation stage. This condition is not identical with the COD criterion since it allows for a variable tip displacement (note that there are no restrictions imposed on  $u(x_p, t)$ , but only the increment, say  $\Delta u_p$ , as defined by the LHS of eq. (A3), is said to be a constant). Of course, when the motion attains the steady-state limit, i.e. when the length of the plastic zone R remains constant and the crack runs fast enough to justify the quasi-steady approximation of Glennie and Willis over the entire plastic zone, both criteria coincide.

Interestingly, the final stretch criterion is identical with McClintock's condition of critical strain attained over the Neuber domain, provided that one defines the strains within the plastic zone of a Dugdale crack as follows

$$(A4) \quad \varepsilon(x_1) = \varepsilon_0 + \frac{\Delta \varepsilon_f^p}{u_0} \{-|\text{grad } u(x_1)|\}, \quad 0 \leq x_1 \leq R$$

here  $\varepsilon_0$  is the yield strain,  $\varepsilon_f^p$  is the plastic component of the strain at fracture and  $u(x_1)$  is the displacement within the Dugdale plastic zone.

To show the equivalence of the final stretch criterion, as given by eq. (A3), and the McClintock critical strain concept, let us consider the case of a general in-plane loading mode (either mode I or II or both applied simultaneously). For such a case the displacement which results from the appropriate boundary value problem formulation is

$$(A5) \quad u(x_1) = \frac{4Y}{\pi E} \left\{ \int [R(x_1)(R(x_1) - x_1)] - \frac{x_1}{2} \ln \left( \frac{\sqrt{R(x_1)} + \sqrt{[R(x_1) - x_1]}}{\sqrt{R(x_1)} - \sqrt{[R(x_1) - x_1]}} \right) \right\}$$

Note that the coordinate  $x_1$  is used here as a time-like parameter. Since the length of the plastic zone is a function of time, it will therefore depend on  $x_1$  in an a-priori unknown fashion. To emphasize this point the symbol  $u(x_1)$  will be replaced by  $u = u(x_1, R(x_1))$ .

To apply the final stretch criterion we have to compute the displacements at the point P at time  $t$  and time  $t - \delta t$ . These two instances correspond to  $x_1 = 0$  and  $x_1 = \Delta$ , respectively. From (A5) we have

$$u(t) = u(x_1=0) = \frac{4Y}{\pi E} R(0) \approx \frac{4Y}{\pi E} \{R(\Delta) - \left(\frac{dR}{dx_1}\right)_{x_1=\Delta} \cdot \Delta\} \quad (A6)$$

$$u(t-\delta t) = u(x_1=\Delta) = \frac{4Y}{\pi E} \left\{ \int [R(\Delta)(R(\Delta)-\Delta)] - \frac{\Delta}{2} \ln \left( \frac{\sqrt{R(\Delta)} + \sqrt{[R(\Delta)-\Delta]}}{\sqrt{R(\Delta)} - \sqrt{[R(\Delta)-\Delta]}} \right) \right\}$$

Applying the fracture criterion (A3) we arrive at

$$(A7) \quad R + \Delta \frac{dR}{d\ell} - \int [R(R-\Delta)] + \frac{\Delta}{2} \ln \frac{\sqrt{R} + \sqrt{[R-\Delta]}}{\sqrt{R} - \sqrt{[R-\Delta]}} = R_0$$

where all  $R$ 's are taken at  $x_1 = \Delta$  (or at the time  $\tau = t - \delta t$ ). This is the governing equation of motion relevant to the slow propagation stage in an elastic-plastic solid, under the in-plane mode of loading (mode I or II).

Equation (A7) describes the universal "R-curve", which was earlier discussed by Rice [6] for mode III. This curve is also equivalent to the "G-curve" introduced by Brown and Srawley in 1964, [1].

Note that the initial slope of the R-curve described by eq. (A7)

$$(A8) \quad \left(\frac{dR}{d\ell}\right)_{\ell=\ell_0} = \sqrt{\left[\frac{R_0}{\Delta} \left(\frac{R_0}{\Delta} - 1\right)\right]} + \frac{1}{2} \ln \left( \frac{\sqrt{R_0/\Delta} - \sqrt{[(R_0/\Delta) - 1]}}{\sqrt{R_0/\Delta} + \sqrt{[(R_0/\Delta) - 1]}} \right)$$

is remarkably similar to the initial slope of the R-curve predicted for mode III by Rice [6]

$$(A9) \quad \left(\frac{dR}{d\ell}\right)_{\ell=\ell_0}^{\text{mode III}} = \frac{R_0}{\Delta} - 1 - \ln (R_0/\Delta)$$

If we introduce a parameter  $\alpha$  (which equals the ratio of the plastic strain at fracture to the yield strain  $\epsilon_f^p/\epsilon_0$ ), then both (A8) and (A9) can be re-written as follows

$$(A10) \quad \left( \frac{dR}{d\ell} \right)_0 = \begin{cases} \sqrt{\alpha(\alpha+1)} + \frac{1}{2} \ln \frac{\sqrt{1+\alpha} - \sqrt{\alpha}}{\sqrt{1+\alpha} + \sqrt{\alpha}} & (a) \\ \alpha - \ln(1+\alpha) & (b) \end{cases}$$

The graphs resulting from eqs. (a) and (b) above, and representing our solution and Rice's are shown in Fig. A1. They converge for both small and large values of the ductility parameter  $\alpha$ .

In fact, within the range of practical applications,  $R$  will be much greater than  $\Delta$ . Equation of motion (A7) reduces then to the simple form

$$(A11) \quad \Delta \frac{dR}{d\ell} = R_0 - \frac{\Delta}{2} \ln \left( \frac{4R}{\Delta} \right)$$

This form results also from McClintock's critical strain criterion of fracture, if one defines the strains as in (A4) and makes use of the condition

$$(A12) \quad \varepsilon(x_1 = \Delta) = \varepsilon_f, \text{ where } \varepsilon_f = \varepsilon_o + \varepsilon_f^p$$

To demonstrate this point let us combine (A4) and (A12)

$$(A13) \quad - \left| \text{grad } u(x_1) \right|_{x_1 = \Delta} = u_o / \Delta$$

and let us compute the gradient of the displacement within the Dugdale plastic zone. Since  $u = u(x_1, R(x_1))$ , we have

$$(A14) \quad \left| \text{grad } u(x_1) \right| = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial R} \frac{dR}{dx_1}$$

where

$$(A15) \quad \frac{\partial u}{\partial R} = \frac{4Y}{\pi E} \frac{1}{2} \ln \left\{ \frac{\sqrt{R(x_1)} - \sqrt{R(x_1) - x_1}}{\sqrt{R(x_1)} + \sqrt{R(x_1) - x_1}} \right\}$$

$$\frac{\partial u}{\partial R} = \frac{4Y}{\pi E} \int \left[ \frac{R(x_1) - x_1}{R(x_1)} \right]$$

Note that  $dR/dx_1$  can be replaced by  $-dR/d\ell$ , since  $x = \ell + x_1 = \text{const.}$

Substituting (A15) and (A14) into the critical strain fracture criterion (A13), and remembering that  $R/\Delta \gg 1$ , we get the equation of motion identical with the equation (A11) which resulted from the final stretch condition.

Eq. (A11), describing the R-curve for an elastic-plastic solid, is a first order linear differential equation and it can be integrated in a closed form. The solution expressed in terms of  $\Delta$ -units is

$$(A16) \quad \ell - \ell_0 = \left(\frac{1}{8}\right) \left[ \exp(2 R_0/\Delta) \right] \left\{ \text{ei} \left[ \ln \left( \frac{4R_0}{\Delta} \right) - \frac{2R_0}{\Delta} \right] - \text{ei} \left[ \ln \left( \frac{4R}{\Delta} \right) - \frac{2R_0}{\Delta} \right] \right\}$$

while in terms of  $R_*$  ( $= \pi K_C^2 / 8Y^2$ ) units it is<sup>+</sup>

$$(A17) \quad \ell - \ell_0 = 2R_* \left\{ \text{ei} \left[ \ln \frac{R_0}{R_*} \right] - \text{ei} \left[ \ln \frac{R}{R_*} \right] \right\}$$

$$\text{ei}(x) = \int_{-\infty}^x [e^t/t] dt$$

Indeed, we observe that the amount of slow growth predicted by eq. (A17) does not depend on either the geometrical configuration or on the initial crack length. It is, therefore, a universal relation analogous to Rice's [6] result for mode III.

+) To allow for the change from the  $\Delta$ -units to the  $R_*$ -units, or from the micro to the macro-units, and to satisfy the boundary condition at the steady-state limit  $R = R_*$ , one has to incorporate another postulate regarding the "opening stretch", namely

$$u_0 = \left(\frac{4Y}{\pi E}\right) \frac{\Delta}{2} \ln(4R_*/\Delta)$$

This transforms the equation (A11) into the simple form

$$\frac{dR}{d\ell} = \frac{1}{2} \ln(R_*/R)$$

B. Derivation of the governing equation of motion for a viscoelastic-plastic solid.

Consider a viscoelastic matrix containing a Dugdale crack and described by the constitutive equations

$$(B1) \quad s_{ij}(t) = \int_{-\infty}^t G_1(t-\tau) \frac{\partial e_{ij}(\tau)}{\partial \tau} d\tau$$

$$s(t) = \int_{-\infty}^t G_2(t-\tau) \frac{\partial e(\tau)}{\partial \tau} d\tau$$

The displacement perpendicular to the crack plane ahead of the crack front is given by

$$(B2) \quad u_y(x,t) = u_y^0(x,t) + \int_{t_0}^t \dot{\Psi}(t-\tau) u_y^0(x,\tau) d\tau$$

where  $u^0(x,t)$  is the associated elastic solution to a given boundary value problem. The normalized creep function  $\Psi(t)$  can be readily related to the relaxation moduli  $G_1(t)$  and  $G_2(t)$ . The lower limit of the hereditary integral in (B2) denotes the time at which a given point  $(x,y=0)$  enters the plastic zone. Consider now a moving crack whose front approaches a stationary control point P located at  $x$ , see Fig. A2. For a general in-plane tensile loading mode equation (B2) reads

$$(B3) \quad u_y(x,t) = \frac{4Y}{\pi E} \left\{ \int [R(\tau)(R(\tau) + \ell(\tau) - x)] + \right.$$

$$+ \frac{\ell(\tau) - x}{2} \ln \left( \frac{\sqrt{R(\tau) + \int [R(\tau) + \ell(\tau) - x]} }{\sqrt{R(\tau) - \int [R(\tau) + \ell(\tau) - x]} } \right) +$$

$$\left. + \int_{t_0}^t \dot{\Psi}(t-\tau) \left\{ \int [R(\tau)(R(\tau) + \ell(\tau) - x)] + \frac{\ell(\tau) - x}{2} \ln \left( \frac{\sqrt{R(\tau) + \int [R(\tau) + \ell(\tau) - x]} }{\sqrt{R(\tau) - \int [R(\tau) + \ell(\tau) - x]} } \right) \right\} d\tau \right\}$$

where  $R$  denotes the current length of the plastic zone and  $x_1$  is the distance measured from the crack tip. To apply the final stretch criterion of fracture we need to evaluate the difference



$$\begin{aligned}
\Delta u_p &= u(x_p, t) - u(x_p, t - \delta t) = \\
&u^0(x_p, t) - u^0(x_p, t - \delta t) + \\
(B4) \quad &\int_{t_0}^t \dot{\Psi}(\tau) u^0(x_p, t - \tau) d\tau - \int_{t_0}^{t-\delta t} \dot{\Psi}(\tau) u^0(x_p, t - \delta t - \tau) d\tau = \\
&\Delta u_p^0 + \int_{t_0}^t [\dot{\Psi}(t - \tau) - \dot{\Psi}(t - \delta t - \tau)] u^0(x_p, \tau) d\tau + \\
&+ \int_{t-\delta t}^t \dot{\Psi}(t - \tau) u^0(x_p, \tau) d\tau
\end{aligned}$$

The first term on the RHS of the above expression,  $\Delta u_p^0$ , is known from the elastic-plastic analysis presented in the preceding section

$$(B5) \quad \Delta u_p^0 = R + \Delta \frac{dR}{d\ell} - \int [R(R-\Delta)] + \frac{\Delta}{2} \ln \left( \frac{\sqrt{R} + \sqrt{R-\Delta}}{\sqrt{R} - \sqrt{R-\Delta}} \right)$$

The second term in the final form of (B4) can be shown to be proportional to the second time-derivative of the creep compliance  $\Psi$  and therefore it will be neglected as only the first derivative of  $\Psi$  is retained in the present analysis. The only restriction imposed is that the change in  $\dot{\Psi}$  due to the shift of argument by  $\delta t$  is small. This means that the function  $\Psi$  does not vary rapidly between time zero and  $\delta t$ , and that within this interval it can be approximated by just the first two terms of the McLaurin expansion

$$(B6) \quad \Psi(\delta t) \approx \Psi(0) + \dot{\Psi}(0) \delta t$$

(of course  $\dot{\Psi}$  may vary within the interval  $(t_0, t)$  but it is roughly constant inside each  $\delta t$  section).

The very last term of (B4) is further reduced as follows

$$(B7) \quad \int_{t-\delta t}^t \dot{\Psi}(t-\tau) u^0(x_p, \tau) d\tau = \frac{4Y}{\pi E} \dot{\Psi}(0) \int_{t-\delta t}^t \left\{ \int [R(\tau)(R(\tau) - \ell(t) + \ell(\tau))] \right. \\ \left. - \frac{\ell(t) - \ell(\tau)}{2} \ln \left( \frac{\sqrt{R(\tau)} + \sqrt{[R(\tau) - \ell(t) + \ell(\tau)]}}{\sqrt{R(\tau)} - \sqrt{[R(\tau) - \ell(t) + \ell(\tau)]}} \right) \right\} d\tau$$

Note that the coordinate  $x_p$  is fixed and equal  $\ell(t)$ . All functions appearing in the integrand of expression (B7), although unknown, can be represented by the following Taylor expansions

$$(B8) \quad \begin{aligned} \ell(t) &= \ell(t-\delta t) + \dot{\ell} \cdot \delta t \\ \ell(\tau) &= \ell(t-\delta t) + \dot{\ell} \cdot (\tau - t + \delta t) \\ R(\tau) &= R(t-\delta t) + \dot{R} \cdot (\tau - t + \delta t) \\ R(t) &= R(t-\delta t) + \dot{R} \cdot \delta t \end{aligned}$$

where both  $\dot{\ell}$  and  $\dot{R}$  are considered constant within the  $\delta t$  interval. Inserting the above expressions into (B7) produces

$$(B9) \quad \int_{t-\delta t}^t \dot{\Psi}(t-\tau) u^0(x_p, \tau) d\tau = \\ \left( \frac{4Y}{\pi E} \right) \dot{\Psi}(0) R(t-\delta t) \int_{t-\delta t}^t \left\{ \int \left[ \rho(\tau) \left( \rho(\tau) - \frac{\dot{\ell} \cdot (t-\tau)}{R(t-\delta t)} \right) \right] - \right. \\ \left. - \frac{\dot{\ell} \cdot (t-\tau)}{2R(t-\delta t)} \ln \left( \frac{\sqrt{\rho(\tau)} + \sqrt{[\rho(\tau) - \dot{\ell} \cdot (t-\tau)/R(t-\delta t)]}}{\sqrt{\rho(\tau)} - \sqrt{[\rho(\tau) - \dot{\ell} \cdot (t-\tau)/R(t-\delta t)]}} \right) \right\} d\tau$$

Here  $\rho(\tau)$  denotes the ratio  $R(\tau)/R(t-\delta t)$ . Combining eqs. (B9), (B5) and (B4) (subject to the final stretch criterion) gives the following equation of motion

$$\begin{aligned}
 R - \int [R(R-\Delta)] + \Delta \frac{dR}{d\ell} + \frac{\Delta}{2} \ln \left( \frac{\int R + \int R-\Delta}{\int R - \int R-\Delta} \right) + \\
 (B10) \quad \frac{\delta \Psi}{\Delta} R^2 \int_0^\varepsilon \left\{ \left[ \rho(s) - s \right] - \frac{s}{2} \ln \left( \frac{\int \rho(s) + \int [\rho(s) - s]}{\int \rho(s) - \int [\rho(s) - s]} \right) \right\} ds = R_0
 \end{aligned}$$

Note that  $\delta \Psi = \Psi(\delta t) - \Psi(0)$  and the current time  $\tau$  has been replaced here by a dimensionless variable

$$(B11) \quad s = \frac{\ell \cdot (t - \tau)}{R(t - \delta t)}$$

The upper limit of the integral in (B10) is now defined as

$$(B12) \quad \varepsilon = \frac{\Delta}{R(t - \delta t)}$$

while the function  $\rho(s)$  is given by

$$(B13) \quad \rho(s) = 1 + (\varepsilon - s) dR/d\ell$$

Summarizing this section we may say that the function  $R=R(\ell)$  is determined by the non-linear integro-differential equation (B10), subject to the initial condition  $R=R_0$  at  $\ell=\ell_0$ . In order to make the problem tractable we shall reduce the equation (B10) to a differential equation. This aim is achieved by observing that for  $R/\Delta \gg 1$  the integrand of the integral  $\int_0^\varepsilon \{ \quad \} ds$  can be expanded into a series

$$(B14) \quad \{ \quad \} = 1 + O(\varepsilon)$$

where  $O(\varepsilon)$  denotes terms containing  $\varepsilon$  ( $=\Delta/R$ ) and being on the higher order of magnitude. Retaining only the first term of (B14), and neglecting  $\Delta$  vs.  $R$  in the second and forth terms of (B10), reduces the governing equation of

motion to the following form

$$(B15) \quad \Delta \frac{dR}{d\ell} + \frac{\Delta}{2} \ln \left( \frac{4R}{\Delta} \right) + \delta\psi \cdot R = R_o$$

This is the desired form of the differential equation which allows for direct determination of the R function. Note that the LHS of eq. (B15) can also be written in a more compact form

$$(B16) \quad \Delta u_p = \Delta u_p^o + u^o(x_p, t - \delta t) \delta\psi$$

Next, the increment  $\delta\psi$  is related to the R function. We have

$$(B17) \quad \delta\psi = \psi(\delta t) - \psi(0) \approx \dot{\psi}(0) \delta t$$

Now, denoting the material rate sensitivity  $\dot{\psi}(0)$  by B, and recalling that  $\delta t = \Delta/\dot{\ell}$ , we may further write

$$(B18) \quad \delta\psi = B\Delta/\dot{\ell} = B\Delta \frac{dt}{d\ell} = B\Delta \frac{dQ}{d\ell} \left( \frac{dQ}{dt} \right)^{-1}$$

But R depends on  $\ell$  in the following way

$$(B19) \quad R = R[\ell, Q(\ell)]$$

where  $Q = Q(\ell)$  is a function describing the applied load (or dimensionless loading parameter). This generates

$$(B20) \quad \frac{dR}{d\ell} = \frac{\partial R}{\partial \ell} + \frac{\partial R}{\partial Q} \frac{dQ}{d\ell}$$

hence

$$(B21) \quad \frac{dQ}{d\ell} = \left( \frac{dR}{d\ell} - \frac{\partial R}{\partial \ell} \right) \left( \frac{\partial R}{\partial Q} \right)^{-1}$$

and

$$(B22) \quad \delta\psi = \frac{B}{\dot{Q}} \Delta \left( \frac{dR}{d\ell} - \frac{\partial R}{\partial \ell} \right) \left( \frac{\partial R}{\partial Q} \right)^{-1}$$

Let us denote the ratio of <sup>the</sup> rate sensitivity B and the rate of loading  $\dot{Q}$  by a letter C, and insert the result (B22) into the equation (B15). This gives our governing equation of motion the following form

$$(B23) \quad \frac{\Delta}{2} \ln\left(\frac{4R}{\Delta}\right) + \Delta[1 + CR (\partial R/\partial Q)^{-1}] \frac{dR}{d\ell} - \\ - \Delta C (\partial R/\partial Q)^{-1} R (\partial R/\partial \ell) = R_o, \quad C = B/\dot{Q}$$

Note that for  $C = 0$ , i.e. for a zero rate-sensitivity  $B$  or an infinite rate of loading  $\dot{Q}$ , we recover the equation of motion valid for an elastic-plastic solid, as shown in the preceding section by eq. (A11).

The other limit case follows from the equation

$$(B24) \quad \Delta u_p^o + u^o(x_p, t - \delta t) \delta \Psi = u_o$$

if one considers a purely visco-elastic solid. There plasticity effects are negligible and one may think of  $\Delta$  and  $R$  as being of the same order of magnitude (in other words the "plastic zone" shrinks to just the "process zone"). Of course  $\Delta$  is still sufficiently small to justify the quasi-steady-state approximation, i.e.  $\dot{\ell} = \text{const.}$  within the time interval of  $\delta t = \Delta/\dot{\ell}$ . Under these conditions the first term of eq. (B24) reduces

$$(B25) \quad \Delta u_p^o = \frac{\Delta/\dot{\ell}}{R/\dot{\ell}} u^o(t - \delta t) = \frac{\Delta}{R} u^o(t - \delta t) = u^o(t - \delta t)$$

This added to the second term gives

$$(B26) \quad (1 + \delta \Psi) u^o(t - \delta t) = u_o$$

or

$$(B27) \quad \Psi(\delta t) R^o = R_o \quad (\text{or } R_c)$$

or

$$\Psi(\delta t) \mathcal{G}^o = \mathcal{G}_o \quad (\text{or } \mathcal{G}_c)$$

The superscript "o" denotes an elastic field entity, while the subscript "o" denotes an initiation level of the same entity (a material constant). Since the propagation occurs here at a certain steady-state value of  $R$ , the initiation and the critical levels can be regarded equal.

Equation (B27) is identical with the Knauss-Dietmann equation given for cracks moving in linearly visco-elastic solids [A7]. A somewhat different result was obtained by Kostrov and Nikitin [A8], but their result can be shown to converge to (B27) if one assumes negligible plastic effects.

C. Comparison of Slow Growth Theories Based on (a) Cherepanov's, (b) Morozov's and (c) Wnuk's Criteria.

(a) Constant Fracture Energy Hypothesis (Cherepanov).

Three hypotheses are discussed here. The first one was proposed by Cherepanov [2]. The basic physical assumption made by Cherepanov is that the total work done in separating two surfaces during an incremental growth is a material constant. This statement expressed in terms of elastic field entities and with the assumption that the Dugdale model applies, reads

$$(C1) \quad 4 \int_{\ell}^a Y \delta u(x, Q(\ell), \ell) dx = 2 \mathcal{G}_c \delta \ell$$

or

$$(C2) \quad 2Y \int_{\ell}^a \left[ \left( \frac{\partial u}{\partial \ell} \right)_Q + \left( \frac{\partial u}{\partial Q} \right)_\ell \frac{dQ}{d\ell} \right] dx = \mathcal{G}_c$$

or

$$(C3) \quad 2Y \left[ \frac{\partial}{\partial \ell} + \frac{dQ}{d\ell} \frac{\partial}{\partial Q} \right] \int_{\ell}^a u[x, Q(\ell), \ell] dx + 2Yu(\text{tip}) = \mathcal{G}_c$$

(compare Wnuk [9]). The above relation describes a slowly moving crack within the subcritical range of the applied load  $Q$ . Symbol  $u$  denotes the displacement perpendicular to the crack surface,  $\ell$  and  $a$  denote the half-length of the crack and the half-length of an extended (Dugdale) crack, respectively. Formula (C1) is valid for an elastic-plastic solid which obeys the Tresca yield condition, and it does not account for the rate sensitivity.

An extension of the Cherepanov theory was proposed for the visco-elastic solids by Wnuk [A3]. The governing equation of motion for such a case is only slightly different from (C1), and it reads

$$(C4) \quad 2Y \left\{ \int_{\ell}^a \left[ \left( \frac{\partial u^0}{\partial \ell} \right)_Q + \left( \frac{\partial u^0}{\partial Q} \right)_\ell \frac{dQ}{d\ell} \right] dx \right\} \Psi(\Delta/\dot{\ell}) = \mathcal{G}_c$$

Here  $u^0$  denotes the associated elastic solution for the same boundary value problem,  $\Psi$  is the normalized creep compliance function,  $\Delta$  is the characteristic dimension of the Neuber domain and  $\dot{\ell}$  the rate of crack growth.

Let us briefly present the essential results pertinent to the small scale yielding range. The integrals involved in (C2) or (C4) are evaluated as follows ( $R/\ell \ll 1$ ):

$$(C5) \quad \int_{\ell}^a \frac{\partial u}{\partial \ell} dx = - \int_{\ell}^a \frac{\partial u}{\partial x} dx = u(\ell) - u(a) = u(\ell) = \left( \frac{4Y}{\pi E} \right) R(\ell, Q(\ell))$$

$$\int_{\ell}^a \frac{\partial u}{\partial Q} dx = \frac{\partial}{\partial Q} \int_{\ell}^a u dx = \frac{\partial}{\partial Q} \left( \frac{4Y}{\pi E} \right) \frac{R^2(\ell, Q(\ell))}{3} = \left( \frac{4Y}{\pi E} \right) \frac{2}{3} R \frac{\partial R}{\partial Q}$$

Combining the above results in accordance with (C2) and (C4) and recalling that  $\mathcal{G}_c$  can be replaced by  $2Yu_*$  or  $2Y(4Y/\pi E)R_*$  we arrive at

$$(C6) \quad R + \frac{2}{3} R \frac{\partial R}{\partial Q} \frac{dQ}{d\ell} = R_*$$

for an elastic-plastic solid, and

$$(C7) \quad \left( R + \frac{2}{3} R \frac{\partial R}{\partial Q} \frac{dQ}{d\ell} \right) \Psi(\Delta/\dot{\ell}) = R_*$$

for a viscoelastic-plastic solid. Note that the rate-sensitivity and the time-dependent properties of a solid are reflected in the latter expression by the creep function  $\Psi$ . Of course this function will depend on the propagation rate  $\dot{\ell}$ . To reduce the above forms to just one equation

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which would contain only the sought-for function  $R=R(\ell)$ , let us eliminate the rates  $dQ/d\ell$  and  $d\ell/dt$ . This is done in two steps. First  $\Psi(\Delta/\dot{\ell})$  is approximated by the first two terms of its McLaurin expansion

$$(C8) \quad \Psi \approx \Psi(0) + \dot{\Psi}(0)\Delta/\dot{\ell}, \quad \Psi(0) = 1, \dot{\Psi}(0) = B$$

and then  $d\ell/dt$  is replaced by  $\dot{Q}(dQ/d\ell)^{-1}$ . Since the  $R$ -function depends explicitly on  $\ell$  and  $Q(\ell)$ , we have also

$$(C9) \quad \frac{dR}{d\ell} = \frac{\partial R}{\partial \ell} + \frac{\partial R}{\partial Q} \frac{dQ}{d\ell}$$

hence

$$(C10) \quad \frac{dQ}{d\ell} = \left( \frac{dR}{d\ell} - \frac{\partial R}{\partial \ell} \right) \left( \frac{\partial R}{\partial Q} \right)^{-1}$$

and

$$(C11) \quad \Psi = 1 + \frac{B\Delta}{\dot{Q}} \left( \frac{dR}{d\ell} - \frac{\partial R}{\partial \ell} \right) \left( \frac{\partial R}{\partial Q} \right)^{-1}$$

Therefore, the governing equation of motion can be written for both cases as

$$(C12) \quad \left\{ R + \frac{2}{3}R \left( \frac{dR}{d\ell} - \frac{\partial R}{\partial \ell} \right) \right\} \cdot \left\{ 1 + C \left( \frac{dR}{d\ell} - \frac{\partial R}{\partial \ell} \right) \left( \frac{\partial R}{\partial Q} \right)^{-1} \right\} = R_*$$

where the parameter  $C = B/\dot{Q}$  describes the time-dependent response of the material. Obviously, when  $C = 0$ , i.e. when either the rate-sensitivity  $B$  is zero or the rate of loading  $\dot{Q}$  becomes infinite, equation (C12) reduces to the equation (C6) which describes propagation in an elastic-plastic, rate-insensitive material.

Let us illustrate applications of eq. (C12) for a case of a central crack contained in a large plate (plane stress only is considered). For this configuration the plastic zone size  $R$  is the following function of the crack length and <sup>the</sup> loading parameter  $\dot{Q}$  ( $= \pi \sigma / 2Y$ ,  $\sigma$  = applied stress,  $Y$  = yield point)

$$(C13) \quad R = \frac{1}{2} \ell Q^2(\ell)$$

Normalizing  $R$  and  $\ell$  by  $R_*$  ( $=\pi K_c^2/8Y^2$ ), and inserting

$$(C14) \quad \frac{\partial R}{\partial Q} = \ell Q \text{ and } \frac{\partial R}{\partial \ell} = \frac{1}{2} Q^2$$

into eq. (C12) produces

$$(C15) \quad \left\{ \rho + \frac{2}{3} \rho \left( \frac{d\rho}{d\zeta} - \frac{\rho}{\zeta} \right) \right\} \cdot \left\{ 1 + C \left( \frac{\Delta}{R_*} \right) \left( \frac{d\rho}{d\zeta} - \frac{\rho}{\zeta} \right) (2\rho\zeta)^{-1/2} \right\} = 1$$

$$(C16) \quad \frac{dQ}{d\zeta} = \frac{2 - Q^2\zeta}{Q^2\zeta \left( \frac{2}{3} Q\zeta + C_0 \right)}$$

Here  $\rho = R/R_*$ ,  $\zeta = \ell/R_*$  and  $C_0 = (B/\dot{Q})(\Delta/R_*)$ . The functions to be determined from the above non-linear first order differential equations are

$$R = R(\ell) \quad \text{from eq. (C15)}$$

and

$$Q = Q(\ell) \quad \text{from eq. (C16)}$$

We do not have an analogous treatment for a large-scale yielding rate-sensitive problem. However, if the rate-sensitivity is absent, the governing equation of motion, i.e. eq. (C2), can be shown to take on this form (central crack configuration):

$$(C17) \quad \frac{dQ}{d\zeta} = \frac{2[1 - \zeta(Q \tan Q + \ln \cos Q)]}{\zeta^2 [Q \sec^2 Q - \tan Q]}$$

for a plane crack, and

$$(C18) \quad \frac{\zeta^2 q^3/3}{(1 - q^2)^{3/2}} \cdot \frac{dq}{d\zeta} + \zeta \frac{1 - (1 - q^2)^{1/2}}{(1 - q^2)^{1/2}} = 1, \quad q = \frac{2}{\pi} Q$$

for a penny-shaped crack. Of course both the above equations reduce to a common form within the small scale yielding range, namely

$$(C19) \quad \frac{dQ}{d\zeta} = \frac{3}{2} \frac{2 - Q^2\zeta}{Q^3\zeta^2}$$



(b) Criterion of Constant COD at Instability (Morozov).

In 1969 Morozov [A2] proposed that the subcritical growth ought to be controlled by the COD criterion rather than a constant specific fracture energy concept. Since in the subcritical range the tip displacement cannot be considered constant (in fact it is a monotonic function of the load remaining in equilibrium with a crack of a given length), one would only require that the final value of the tip displacement, attained at the terminal instability point, should be a material constant. This is exactly what Morozov suggested. In order to ensure a constant COD at the end point of the stable growth, he modifies the fracture energy  $\mathcal{G}_c$  in such a way that the terminal instability is always reached in accordance with the COD criterion. Thus Morozov's  $\mathcal{G}_c$  becomes a function of geometry and the current crack length. Let us denote this function by  $\mathcal{G}_R$ , where the letter "R" stands for "resistance". A similar concept was proposed in 1968 by Wnuk [A4] and [A5].

The function  $\mathcal{G}_R$  is evaluated from the following equation

$$\begin{aligned} \mathcal{G}_R(l) &= 2Y \int_l^a \frac{\partial u(x, l)}{\partial l} dx \\ (C20) \quad &= 2Y \left\{ u(l, l) + \frac{\partial}{\partial l} \int_l^a u(x, l) dx \right\} \quad Q = Q_{crit} \end{aligned}$$

The expression contained in the bracket becomes a function of  $l$  only, since  $Q$  is eliminated through the use of the COD criterion. (This latter criterion is applied to evaluate  $Q_{crit}$  first). The next step, according to Morozov, is to apply the energy balance equation which reads simply

$$(C21) \quad 2 \int_l^a Y \delta u(x, Q(l), l) dx = \mathcal{G}_R(l) \delta l$$

Note that his operator  $\delta$  implies an identical operation as the one indicated in eq. (C2).

To summarize briefly Morozov's theory let us write down the essential results. First, the function  $\mathcal{G}_R(\ell)$  for a plane crack is found to be

$$(C22) \quad \mathcal{G}_R = \mathcal{G}_c \{ [1 - \exp(-2/\zeta)]^{1/2} \zeta \exp(1/\zeta) \cos^{-1} [\exp(-1/\zeta)] - 1 \}$$

and for a penny-shaped crack (compare Wnuk [A4] and [5]) it is

$$(C23) \quad \mathcal{G}_R = \mathcal{G}_c \left\{ \frac{\zeta}{\zeta - 1} \right\}$$

Note that both expressions reduce to just Irwin's  $\mathcal{G}_c$  for a sufficiently long crack ( $\zeta \gg 1$ ). Combining the above formulae with Morozov's criterion for crack extension (C21) gives

$$(C24) \quad \frac{dQ}{d\zeta} = \frac{2\{F(\zeta) - \zeta(Q \tan Q + \ln \cos Q)\}}{\zeta^2 [Q \sec^2 Q]}$$

for a 2D crack, and

$$(C25) \quad \frac{dq}{d\zeta} = \frac{3(1-q^2)^{3/2} \{ G(\zeta) - \zeta[1 - (1-q^2)^{1/2}] (1-q^2)^{-1/2} \}}{q\zeta^2}, \quad q = \frac{2}{\pi} Q$$

for a penny-shaped crack. The geometry dependent functions  $F(\zeta)$  and  $G(\zeta)$  are given by the expressions inside the brackets of (C22) and (C23), respectively. No rate effects are included in either of the above equations. When the plasticity effects are negligible ( $\zeta \gg 1$  or  $Q \rightarrow 0$ ), one can show that both the equations (C24) and (C25) reduce to a common form

$$(C26) \quad \frac{dQ}{d\zeta} = \frac{2 - Q^2}{\zeta^2 Q}$$

This equation, surprisingly, has a closed form solution which was not mentioned by Morozov, and it reads

$$(C27) \quad Q(\zeta) = \zeta^{-1} [4(\zeta - \zeta_0) + \zeta_0^2 Q_0^2]^{1/2}$$

Obviously, the initial condition  $Q(\zeta_0) = Q_0$  is included in the above expression. Due to the simplicity of eq. (C27) we may directly compute the load and the crack length at the terminal instability point. Equating  $dQ/d\zeta$  to zero we get

$$(C28) \quad Q_{\text{crit}} = \sqrt{2/\zeta_{\text{crit}}}$$

$$\ell_{\text{crit}} = \ell_0 [2 - R_0/R_*] \quad , \quad R_0 = \frac{1}{2} \ell_0 Q_0^2$$

where  $R_0$  is the plastic zone size at the onset of slow propagation and  $R_*$  is the value of  $R$  at the fully developed steady-state growth. Formula (C28) appears to have some experimental support, cf. Sullivan and Freed [A6].

We may add that somewhat unexpectedly the governing equation (C26) may also be derived directly from the classical Griffith energy balance equation

$$(C29) \quad \delta U = 4Y\delta\ell$$

if the operator  $\delta$  is applied in the following sense

$$(C30) \quad \delta U = \left\{ \frac{\partial}{\partial \ell} U + \frac{\partial U}{\partial Q} \frac{dQ}{d\ell} \right\} \delta\ell$$

With  $U = \pi \ell^2 \sigma^2/E$ ,  $Q = \pi \sigma/2Y$  and  $R_* = \pi K_c^2/8Y^2$ , the energy balance equation (C29) supplemented by (C30) reduces to eq. (C26).

To conclude the section on Morozov's criterion of fracture, we add visco-elastic properties into his model. For this purpose consider the small scale yielding range only, and apply Wnuk's [9] result for the "effective" strain energy due to a crack contained in a visco-elastic solid

$$(C31) \quad U_{\text{eff}} = U^0 \Psi(\Delta/\dot{\ell})$$

Here, the associated elastic strain energy  $U^0$  is  $(1/2)R_* \mathcal{G}_c Q^2 \zeta^2$ , and the function  $\Psi$  can be approximated by  $1 + \dot{\Psi}(0) \Delta/\dot{\ell}$ , while  $\dot{\ell}$  is replaced by  $R_* \dot{Q}(dQ/d\zeta)^{-1}$ . Then, the balance of energy criterion

$$(C32) \quad \Psi(\Delta/\dot{\ell}) \delta U^0 = 2 \mathcal{G}_c \delta \ell$$

becomes

$$(C33) \quad \left(1 + \frac{B\Delta}{\dot{Q}R_*} \frac{dQ}{d\zeta}\right) \cdot (Q^2 \zeta + Q \zeta^2 \frac{dQ}{d\zeta}) = 2$$

This is a non-linear differential equation defining  $dQ/d\zeta$  in terms of  $Q$  and  $\zeta$ , so that the numerical integration by Runge-Kutta method presents no problem. If we omit\*, however, the terms containing  $(dQ/d\zeta)^2$ , then the above equation simplifies to a form remarkably similar to eq. (C26), namely

$$(C34) \quad \frac{dQ}{d\zeta} = \frac{2 - \zeta Q^2}{\zeta Q [C_0 Q + \zeta]}$$

Here, the constant  $C_0$  encompasses the following group of parameters

$$C_0 = \frac{B}{\dot{Q}} \cdot \frac{\Delta}{R_*}$$

The ratio  $\Delta/R_*$  can be roughly approximated by the quotient  $R_0/R_*$ , where  $R_0$  denotes the threshold value of the plastic zone size. Obviously, with  $C_0=0$ , we recover the rate-insensitive equation (C26).

\* The test run on IBM 360 shows no appreciable difference in the shape of the integral curve resulting from the complete equation (C33) and the simplified one (C34).

(c) Final Stretch Criterion (Wnuk)

The criterion of fracture proposed in this work and in [8] requires that the amount of deformation which occurs within the process zone during the time interval just prior to fracture is a material property. The deformation involved here is identified with the increment of displacement  $\Delta u_p$  produced at the control point P, within the time  $\delta t = \Delta/\dot{l}$ . Thus we have

$$(C35) \quad \Delta u_p = u(x_p, t) - u(x_p, t - \delta t) = R_0 (4Y/\pi E)$$

where the right hand side contains the material constants only. If the viscoelastic displacements are substituted into the above expression and R is considered small vs. the crack size, the equation (C35) takes on the form (compare eq. (B23)):

$$(C36) \quad \frac{dR}{d\ell} = \frac{R_0 - (1/2)\ell n(4R/\Delta) + R(\partial R/\partial \ell) C\Delta \cdot (\partial R/\partial Q)^{-1}}{\Delta + R C\Delta (\partial R/\partial Q)^{-1}}$$

Here  $C = B/\dot{Q}$ , and  $B (= \dot{\psi}(0))$  denotes the rate sensitivity. In order to adjust the right hand side of (C35) to the boundary condition at  $R(\ell=\ell_{crit}) = R_*$ , we have made an additional assumption regarding  $R_0$ , namely

$$(C36a) \quad R_0 = \frac{\Delta}{2} \ell n(4R_*/\Delta)$$

With this substitution and upon normalizing both R and  $\ell$  by the steady-state value of the plastic zone size  $R_*$ , eq. (C35) becomes

$$(C37) \quad \frac{d\rho}{d\zeta} = \frac{\frac{1}{2}\ell n(1/\rho) + C\rho(\partial\rho/\partial\zeta)(\partial\rho/\partial Q)^{-1}}{1 + C\rho(\partial\rho/\partial Q)^{-1}}$$

$$\rho = R/R_*, \quad \zeta = \ell/R_*$$

To apply this equation for a certain crack configuration, one has to supplement it with the function  $R(Q, \ell)$ , which is known from the stress analysis of the corresponding Dugdale crack problem. Let us illustrate this point for the simplest case of a central crack in an infinite plate under tension  $\sigma$ .

We have here

$$(C38) \quad R = \frac{1}{2} \ell (\pi \epsilon / 2Y)^2, \text{ or}$$

$$\rho = \frac{1}{2} \zeta Q^2, \quad Q = \pi \epsilon / 2Y$$

Hence we find the derivatives required in (C37)

$$(C39) \quad \frac{\partial \rho}{\partial Q} = \sqrt{2\rho\zeta}, \quad \frac{\partial \rho}{\partial \zeta} = \rho/\zeta$$

and we have

$$(C40) \quad \frac{d\rho}{d\zeta} = \frac{\frac{1}{2} \ln(1/\rho) + C_0^2/\zeta \sqrt{2\zeta\rho}}{1 + C_0/\sqrt{2\zeta\rho}}$$

as the governing equation of the problem. It can be readily transformed into the  $(Q, \zeta)$  plane. With the use of (C38) we obtain

$$(C41) \quad \frac{dQ}{d\zeta} = \frac{\ln(2/Q^2\zeta) - Q^2}{Q\zeta(2 + CQ)}$$

This is integrated numerically for a chosen set of the initial crack length  $\zeta_0$  and the initiation (or threshold) load  $Q_0$ . The curves generated in this way, for various rate-sensitivities  $C$ , are then compared with those which resulted from Cherepanov's and Morozov's hypotheses. The graphs are shown in Fig. A3. One may observe rather pronounced differences between the three theories described here, especially for the parameter  $C$  approaching zero (i.e. fast loading, or negligible rate-sensitivity). For large  $C$  all three curves seem to converge. In the limit of infinite  $C$  the graphs reduce to a horizontal line, which corresponds to a "creeping crack", described in more detail in [9].

D. "Resistance" Function or Apparent Surface Energy.

A brittle solid fractures when the elastic energy release rate  $\mathcal{G}$  attains the critical level, say  $\mathcal{G}_c$ , which in turn is related to the material toughness and the Young modulus ( $\mathcal{G}_c = K_c^2/E$ ). The Griffith-Irwin criterion for brittle fracture simply requires that the equality  $\mathcal{G} = \mathcal{G}_c$  holds.

This simple view complicates a little when we try to describe fracture in inelastic solids such as elastic-plastic, or viscoelastic-plastic materials. There the criterion for fracture (equation of motion) reads \*

$$(D1) \quad R \left\{ 1 + \frac{2}{3} \frac{\partial R}{\partial Q} \frac{dQ}{d\ell} \right\} \Psi(\Delta/\dot{\ell}) = R_*$$

This equation can be cast into a form resembling closely the classical criterion  $\mathcal{G} = \mathcal{G}_c$ , if we agree to modify its right hand side. Note first that the expression  $(R/R_*) \mathcal{G}_c$  is identical with the elastic energy release rate  $\mathcal{G}$ . Now we can re-write (D1) as follows

$$(D2) \quad \mathcal{G} \left\{ 1 + \frac{2}{3} \frac{\partial R}{\partial Q} \frac{dQ}{d\ell} \right\} \Psi(\Delta/\dot{\ell}) = \mathcal{G}_c$$

or, finally

$$(D3) \quad \mathcal{G} = \mathcal{G}_R$$

where the new function  $\mathcal{G}_R$  is defined by (D2) and (D3)

$$(D4) \quad \mathcal{G}_R = \mathcal{G}_c \left\{ \left[ 1 + \frac{2}{3} \frac{\partial R}{\partial Q} \frac{dQ}{d\ell} \right] \Psi(\Delta/\dot{\ell}) \right\}^{-1}$$

\*) To focus the attention we consider here only the results based on Cherepanov's hypothesis, therefore eq. (D1) is identical with equation (C7).

The function  $\mathcal{G}_R$  will be referred to as the "resistance" function, or as the "apparent surface energy." For an ideally brittle solid  $\mathcal{G}_R \equiv \mathcal{G}_c$ , but otherwise it is a geometry-dependent and a rate-dependent entity. We believe that an experiment designed to measure the fracture toughness will in fact record the "apparent toughness", related to  $\mathcal{G}_R$  by the well-known formula  $K_R = (E \mathcal{G}_R)^{1/2}$ .

To illustrate possible applications of the proposed equation (D4), let us consider a central crack contained in a large sheet subjected to a remotely applied tension  $\sigma$ . Let us consider first an elastic-plastic solid. The length of the plastic zone is then  $R = (1/2)\ell Q^2$ , where  $Q = \pi\sigma/2Y$  and the derivative  $dQ/d\ell$  can be found from the eq. (C16)

$$(D5) \quad \frac{dQ}{d\ell} = \left\{ \frac{3}{2} \frac{2 - Q^2 \zeta}{Q^3 \zeta^2} \right\} R_*^{-1}, \quad \zeta = \ell/R_*$$

(the rate sensitivity is zero for an elastic-plastic solid while  $\Psi = 1$ ).

Inserting (D5) into (D4) gives the resistance function

$$(D6) \quad \mathcal{G}_R = \mathcal{G}_c \frac{1}{2} Q^2(\zeta) \cdot \zeta$$

where the function  $Q = Q(\zeta)$  has to be determined by integration of (D5).

The curves  $\mathcal{G}_R$  vs.  $\ell$  will depend on the initial crack length and the threshold level of the loading parameter  $Q$ , as shown in Fig. A4. A similar equation for a crack opened by a pair of wedging forces is discussed in [A3]. For this configuration the equation analogous to (D5) is

$$(D7) \quad \frac{dQ}{d\ell} = R_*^{-1} \frac{3\pi^2 \zeta}{2Q} \left[ \frac{2\pi^2 \zeta}{Q^2} - 1 \right]$$

$$Q = \pi P/2YR_* \quad , \quad R = (R_*/2\pi^2) Q^2/\zeta$$



while an analogue of eq. (D6) is

$$(D8) \quad \mathcal{G}_R = \mathcal{G}_c Q^2 / 2\pi^2 \zeta$$

To conclude this section let us take a look at a visco-elastic solid described by a Voigt element ( $E_2, \tau_2$ ) connected in series with a spring  $E_1$ . If all plasticity effects are neglected, equation (D4) gives

$$(D9) \quad \mathcal{G}_R = \mathcal{G}_c / \Psi(\Delta/\dot{l})$$

which for our model will read

$$(D10) \quad \mathcal{G}_R = \mathcal{G}_c \left\{ 1 + \frac{E_1}{E_2} [1 - \exp(-\Delta/\dot{l}\tau_2)] \right\}^{-1}$$

This defines the rate dependence of the apparent  $\mathcal{G}_c$ , often measured in experiments which involve a crack propagating through a visco-elastic medium. The graph constructed according to (D10) is shown in Fig. A5a.

The above relation can be generalized for any visco-elastic solid, if instead of  $\mathcal{G}_R$  vs. the propagation rate one plots  $\mathcal{G}_R$  vs. the current crack length. Then, for a central crack configuration, it may be shown that  $\Psi(\Delta/\dot{l}) = (\sigma_{\text{Griffith}} / \sigma_{\text{applied}})^2 (l_0/l)$ . The resistance function is proportional to the reciprocal of  $\Psi(\Delta/\dot{l})$ , therefore

$$(D11) \quad \mathcal{G}_R = \mathcal{G}_c n \left( \frac{l}{l_0} \right)$$

where  $n$  is the square of the ratio of the applied stress to the short-time Griffith stress. A graph resulting from (D11) is shown in Fig. A5b.

Part II. Dynamic Crack Represented by the Dugdale Model

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SYMBOLS

- $c_1$  = Dilatational wave speed
- $c_2$  = Shear wave speed
- $k = c_2/c_1$
- $c_s$  = Rayleigh wave speed
- $k_s = c_s/c_1$
- $\rho$  = mass density
- $\mu$  = Shear Modulus
- $\nu$  = Poissons Ratio
- $Y$  = Yield stress of material

# ATKINSON-BROBERG-DUGDALE MODEL

## STATEMENT OF PROBLEM

A crack is expanding in an elastic solid under the influence of an applied tensile stress  $P$  at infinity. As shown in Fig. 1 a thin region ahead of each crack tip is deformed plastically, i.e., the stress in this region is assumed to be equal to the yield stress  $Y$  of the material. The length of the crack at a given instant is  $2a$  and the length of each plastic zone is  $w = c - a$ . The crack tips are moving at a constant velocity  $\dot{a}$  and the ends of the plastic zones are moving with velocity  $\dot{c}$ .

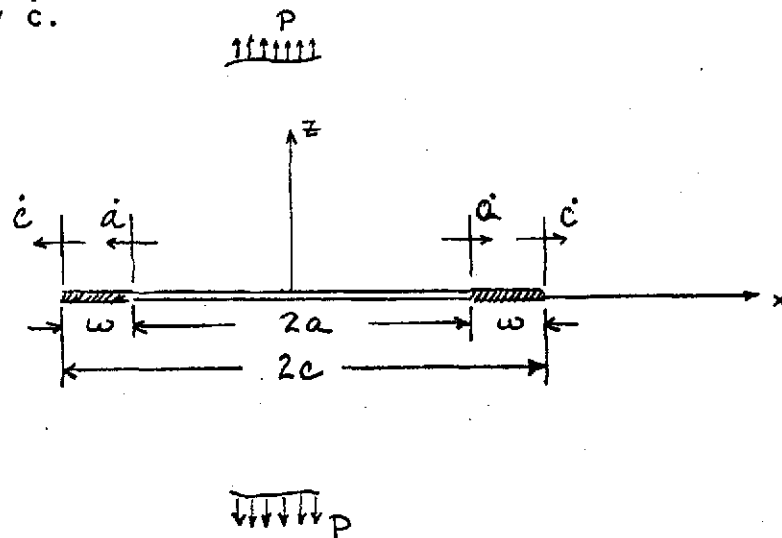


Fig. 1

The solution may be obtained by super-posing the problem of a plate with no crack under uniform tensile stress  $P$  and a crack expanding under the influence of pressure  $P$  over the portion

of the crack surface defined by  $|x| < a$  and a pressure  $(P-Y)$  over the surface  $a < |x| < c$ . Since the problem possesses symmetry with respect to the  $x$ - $y$  plane an equivalent problem for the upper half-space which shall be denoted as Prob. A may be formulated (see Fig. 2). Retaining consistency with Atkinsons solution the formulation is for plane strain. The plane stress solution differs only by a constant and will be obtained from the plane strain solution.

#### PROBLEM A

A pressure  $P$  is acting on the infinite strip  $|x| < at$  on the surface  $z = 0$  of a semi-infinite solid  $z > 0$ . A pressure  $(P-Y)$  is acting on the infinite strips  $at < |x| < ct$  of the surface  $z = 0$ . The vertical displacement  $u_z$  at  $z = 0$  is zero for  $|x| > ct$ .

Find the normal stress at the surface  $z = 0$  for  $|x| > ct$  and the displacement  $u_z$  for  $|x| < ct$ .

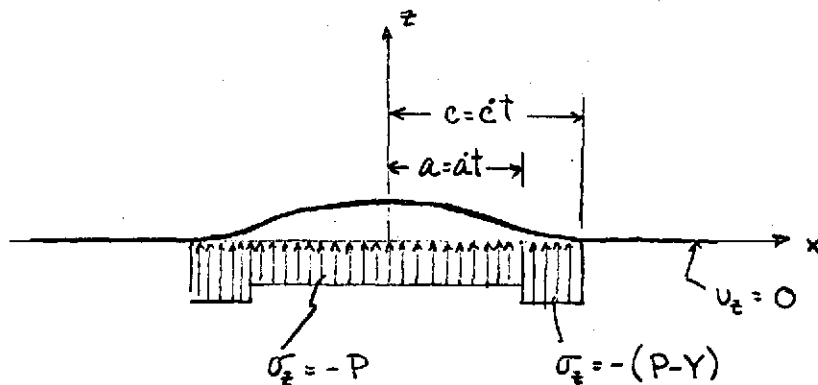


Fig. 2

In order to solve Problem A, the following problem is first solved (see Fig.3).

### PROBLEM B

A pressure  $P$  is acting on the infinite strip  $|x| < at$  of the surface  $z = 0$ . A pressure  $q$  acts on the strips  $at < |x| < ct$  whereas no pressure acts on the surface  $z = 0$  for  $|x| > ct$ . Find the displacement  $u_z$  at the surface  $z = 0$ .

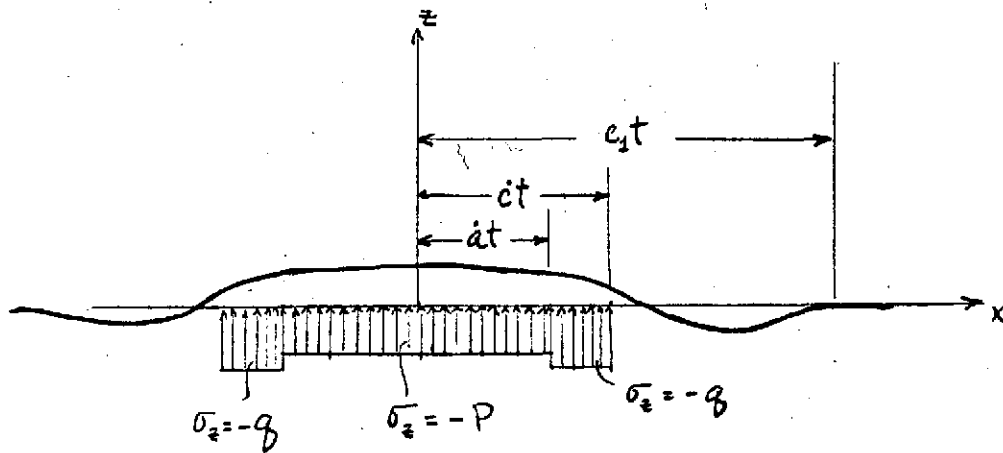


Fig.3

In Fig.3,  $c_1$  is the dilatational wave velocity.

### SOLUTION TO PROBLEM B

The boundary conditions on  $z = 0$  are

$$\begin{aligned} \sigma_z &= -P & 0 < |x| < at \\ &= -q & at < |x| < ct \\ &= 0 & |x| > ct \end{aligned} \quad (1)$$

For the problem where a pressure  $p^*$  is applied to the strip  $0 < |x| \leq \dot{a}_1 t$  and no pressure is applied for  $|x| > \dot{a}_1 t$  the boundary conditions are

$$\begin{aligned} \sigma_z &= -p^* & 0 < |x| \leq \dot{a}_1 t \\ &= 0 & |x| > \dot{a}_1 t \end{aligned} \quad (2)$$

and the following expression has been obtained by Broberg for  $\partial^2 u_z / \partial t^2$  on the surface  $z = 0$  (see Brobergs eq.(16) where his symbols  $q$ ,  $\beta$ ,  $\tau$  are equivalent to  $p^*$ ,  $\dot{a}_1/c_1$ , and  $c_1 t$  in this paper),

$$\begin{aligned} \frac{\partial^2 u_z}{\partial t^2} = & -\frac{p^*}{2\pi k^4 c_1^2 |x|} \left\{ \frac{\dot{a}_1 c_1}{1 - \frac{\dot{a}_1^2 t^2}{x^2}} \operatorname{Re} \left[ F\left(\frac{c_1^2 t^2}{x^2}\right) \right] \right. \\ & \left. + \frac{\pi |x|}{2 t} \dot{a}_1 \operatorname{Im} \left[ F\left(\frac{c_1^2}{\dot{a}_1^2}\right) \right] \delta\left(\frac{\dot{a}_1}{c_1} - \frac{|x|}{c_1 t}\right) \right\} \end{aligned} \quad (3)$$

where

$$F(\theta^2) = \frac{\sqrt{\theta^2 - 1}}{\left(\frac{1}{2k^2} - \theta^2\right)^2 + \theta^2 \sqrt{(1 - \theta^2)\left(\frac{1}{k^2} - \theta^2\right)}} \quad (4)$$

An expression for  $\partial^2 u_z / \partial t^2$  that corresponds to the boundary conditions of eq.(1) may be obtained by superposing three problems: positive pressure  $P$  acting on a strip expanding with velocity  $\dot{a}$ , positive pressure  $q$  acting on a strip expanding with velocity  $\dot{c}$ , and negative pressure  $q$  acting on a strip expanding with velocity  $\dot{a}$ . This is shown in Fig. 4.

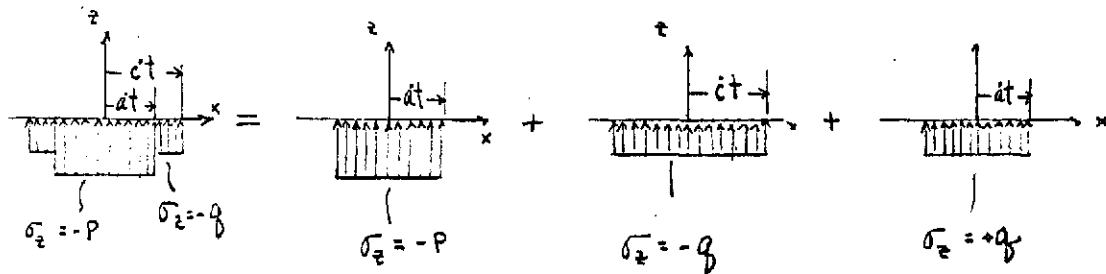


Fig. 4

The result is

$$\frac{\partial^2 u_z}{\partial t^2} = -\frac{q}{2\pi k^4 c^2 |x|} \left\{ \frac{c_1 c_2}{1 - \frac{c_1^2 t^2}{x^2}} \operatorname{Re} \left[ F\left(\frac{c_1^2 t^2}{x^2}\right) \right] + \frac{\pi |x|}{2t} c_1 \operatorname{Im} \left[ F\left(\frac{c_1^2}{c_2^2}\right) \right] \delta\left(\frac{c_1}{c_2} - \frac{|x|}{c_1 t}\right) \right\}$$

$$- \frac{P - q}{2\pi k^4 c^2 |x|} \left\{ \frac{a c_1}{1 - \frac{a^2 t^2}{x^2}} \operatorname{Re} \left[ F\left(\frac{c_1^2 t^2}{x^2}\right) \right] + \frac{\pi |x|}{2t} a \operatorname{Im} \left[ F\left(\frac{c_1^2}{a^2}\right) \right] \delta\left(\frac{a}{c_1} - \frac{|x|}{c_1 t}\right) \right\}, \quad z = 0. \quad (5)$$

#### SOLUTION TO PROBLEM A

Now use the solution of problem B to set up an integral equation solution to problem A. First consider an incremental load on the half space defined by the following boundary conditions (see Fig.5)

$$\begin{aligned} d\sigma_z &= 0 & |x| < ct \\ &= -q'(v)dv & ct < |x| < vt \\ &= 0 & vt < |x| \end{aligned} \quad (6)$$

where,

$$c' \leq v \leq c_1 \quad (7)$$

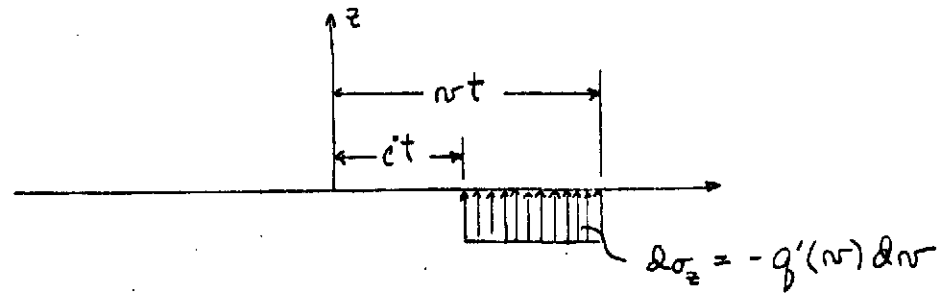


Fig. 5

Making use of eqs.(1) and (5), with appropriate change in variables, the solution for  $\partial^2 u_z / \partial t^2$  corresponding to the boundary conditions of eq.(6) is:

$$\begin{aligned} \frac{\partial^2 u_z}{\partial t^2} = & - \frac{q'(r) dr}{2\pi k^4 c^2 |x|} \left\{ \left[ \frac{r c_1}{1 - \frac{r^2 t^2}{x^2}} - \frac{c c_1}{1 - \frac{c^2 t^2}{x^2}} \right] \operatorname{Re} \left[ F \left( \frac{c_1^2 t^2}{x^2} \right) \right] \right. \\ & + \frac{\pi |x|}{2t} r \operatorname{Im} \left[ F \left( \frac{c^2}{r^2} \right) \right] \delta \left( \frac{r}{c} - \frac{|x|}{c_1 t} \right) \\ & \left. - \frac{\pi |x|}{2t} c \operatorname{Im} \left[ F \left( \frac{c^2}{c_1^2} \right) \right] \delta \left( \frac{c}{c_1} - \frac{|x|}{c_1 t} \right) \right\} \end{aligned} \quad (8)$$

In order to satisfy the stress and displacement boundary conditions for Problem A, superpose stresses  $d\sigma_z$  over the range of eq.(7) together with the stresses

$$\sigma_z = \begin{cases} -P & 0 < |x| < at \\ -q_0 & at < |x| < ct \\ 0 & ct < |x| \end{cases} \quad (9)$$

where  $q_0 = P - Y$ , on the surface  $z = 0$  in such a manner that  $\partial^2 u_z / \partial t^2 = 0$  for  $|x| > ct$ .



The superposed expression for  $\partial^2 u_z / \partial t^2$  is obtained by adding (3) with  $q=q_0$  and the integral over the range  $\dot{c} \leq v \leq c_1$  of eq.(8). Then

$$\begin{aligned} \frac{\partial^2 u_z}{\partial t^2} = & \frac{-q_0}{2\pi k^4 e c_1^2 |x|} \left\{ \frac{c/c_1}{(1 - \frac{c^2 t^2}{x^2})} \operatorname{Re} \left[ F \left( \frac{c^2 t^2}{x^2} \right) \right] \right\} \\ & - \frac{(P-q_0)}{2\pi k^4 e c_1^2 |x|} \left\{ \frac{a/c_1}{(1 - \frac{a^2 t^2}{x^2})} \operatorname{Re} \left[ F \left( \frac{a^2 t^2}{x^2} \right) \right] \right\} \\ & - \frac{\operatorname{Re} \left[ F \left( \frac{c^2 t^2}{x^2} \right) \right]}{2\pi k^4 e c_1^2 |x|} \int_{\dot{c}}^{c_1} \left[ \frac{v c_1}{1 - \frac{v^2 t^2}{x^2}} - \frac{c/c_1}{1 - \frac{c^2 t^2}{x^2}} \right] q'(v) dv \\ & - \frac{1}{4 k^4 e c_1^2 |x|} \int_{\dot{c}}^{c_1} \delta \left( \frac{v}{c_1} - \frac{|x|}{c_1 t} \right) \frac{|x|}{t} v \operatorname{Im} \left[ F \left( \frac{v^2}{x^2} \right) \right] q'(v) dv \end{aligned} \quad (10)$$

$z = 0$

Setting  $\partial^2 u_z / \partial t^2 = 0$  for  $|x| \geq \dot{c}t$  and making the following transformation of variables,

$$\frac{x^2}{c_1^2 t^2} = r, \quad \frac{v^2}{c_1^2} = s, \quad \frac{1}{c_1} q'(v) = \phi(s), \quad (11)$$

we obtain, for  $\dot{c}t < |x| < c_1$ , or equivalently for  $(\dot{c}/c_1)^2 < r < 1$ ,

$$\begin{aligned} 0 = & q_0 \left\{ \frac{(c/c_1)}{(r - c^2/c_1^2)} \operatorname{Re} \left[ F \left( \frac{1}{r} \right) \right] \right\} + (P-q_0) \left\{ \frac{(a/c_1)}{(r - a^2/c_1^2)} \operatorname{Re} \left[ F \left( \frac{1}{r} \right) \right] \right\} \\ & + \frac{1}{2} \operatorname{Re} \left[ F \left( \frac{1}{r} \right) \right] \int_{\frac{c^2}{c_1^2}}^1 \left[ \frac{1}{r-s} - \frac{(c/c_1)}{\sqrt{s}(r - \frac{c^2}{c_1^2})} \right] \phi(s) ds \\ & + \frac{\pi}{2} \phi(r) \operatorname{Im} \left[ F \left( \frac{1}{r} \right) \right], \quad \frac{c^2}{c_1^2} < r < 1 \end{aligned} \quad (12)$$

or rearranging terms,

$$\begin{aligned}
 & -\pi \frac{\text{Im}[F(1/r)]}{\text{Re}[F(1/r)]} \left(r - \frac{c^2}{c_1^2}\right) \phi(r) + \int_{\frac{c^2}{c_1^2}}^1 \frac{\phi(s) \left[ \left(r - \frac{c^2}{c_1^2}\right) \sqrt{s} - \frac{c^2}{c_1} (r-s) \right]}{(s-r) \sqrt{s}} ds \\
 & = \frac{2c^2 q_0}{c_1} + 2(P-q_0) \frac{a \left(r - \frac{c^2}{c_1^2}\right)}{c_1 \left(r - \frac{a^2}{c_1^2}\right)}
 \end{aligned}$$

or

$$\begin{aligned}
 & -\frac{\text{Im}[F(1/r)]}{\text{Re}[F(1/r)]} \left(r - \frac{c^2}{c_1^2}\right) \phi(r) + \frac{1}{\pi} \int_{\frac{c^2}{c_1^2}}^1 \frac{\left(s - \frac{c^2}{c_1^2}\right)}{s-r} \phi(s) ds \\
 & - \frac{1}{\pi} \int_{\frac{c^2}{c_1^2}}^1 \frac{\left(s - \frac{c^2}{c_1^2}\right)}{\sqrt{s}} \phi(s) ds = \frac{2c^2 q_0}{\pi c_1} + \frac{2(P-q_0) a \left(r - \frac{c^2}{c_1^2}\right)}{\pi c_1 \left(r - \frac{a^2}{c_1^2}\right)}, \quad (13) \\
 & \frac{c^2}{c_1^2} < r < 1.
 \end{aligned}$$

Eq. (13) is a singular integral equation which may be written

$$a(r) \phi(r) + \frac{1}{\pi} \int_{\frac{c^2}{c_1^2}}^1 \frac{b(s) \phi(s)}{s-r} ds = q(r), \quad \frac{c^2}{c_1^2} < r < 1 \quad (14)$$

where

$$a(r) = \frac{\text{Im}[F(1/r)]}{\text{Re}[F(1/r)]} \left(r - \frac{c^2}{c_1^2}\right) = \begin{cases} 0, & \frac{c^2}{c_1^2} < r < k^2 \\ \frac{4k^3 \sqrt{(1-r)(r-k^2)}}{(r-2k^2)^2} \left(r - \frac{c^2}{c_1^2}\right), & k^2 < r < 1 \end{cases} \quad (15)$$

$$b(r) = r - \frac{c^2}{c_1^2} \quad (16)$$

$$q(r) = \frac{2q_0 c^2}{\pi c_1} + \frac{1}{\pi} \int_{\frac{c^2}{c_1^2}}^1 \frac{\sqrt{s - \frac{c^2}{c_1^2}}}{\sqrt{s}} \phi(s) ds + \frac{2(P-q_0) a \left(r - \frac{c^2}{c_1^2}\right)}{\pi c_1 \left(r - \frac{a^2}{c_1^2}\right)} \quad (17)$$

The solution of the integral equation (14) may be written in the form (see page 331, Muskhelishvili, Singular Integral Equations)

$$\begin{aligned}
 \text{I. } \phi(r) = & \frac{q(r) a(r)}{a^2(r) + b^2(r)} - \frac{\exp(-\omega(r))}{\pi(r - \frac{c_1^2}{c_2^2}) \sqrt{a^2(r) + b^2(r)}} \int_{\frac{c_1^2}{c_2^2}}^1 \frac{(s^2 - \frac{c_1^2}{c_2^2}) \exp(\omega(s)) b(s) q(s)}{(s-r) \sqrt{a^2(s) + b^2(s)}} ds \\
 & + \frac{L \exp(-\omega(r))}{(r - \frac{c_1^2}{c_2^2}) \sqrt{a^2(r) + b^2(r)}} \quad (18)
 \end{aligned}$$

where  $L$  is an arbitrary constant and

$$\omega(r) = \int_{(c_1^2/c_2^2)}^1 \frac{\oplus(s)}{s-r} ds \quad (19)$$

$$\oplus(r) = \frac{1}{2\pi i} \ln \left( \frac{a(r) + ib(r)}{a(r) - ib(r)} \right) \quad (20)$$

$\omega(r)$  is calculated in Appendix I of Broberg and thus,  $\exp[-\omega(r)]$  is written

$$\exp[-\omega(r)] = \begin{cases} \sqrt{\frac{(1-r)}{(r - \frac{c_1^2}{c_2^2})}} \frac{r(r - k^2)}{(r - 2k^2)^2 - 4k^3 \sqrt{(1-r)(k^2 - r)}} , & \frac{c_1^2}{c_2^2} < r < k^2 \\ \sqrt{\frac{(1-r)}{(r - \frac{c_1^2}{c_2^2})}} \frac{r(r - k^2)}{\sqrt{(r - 2k^2)^4 + 16k^6(1-r)(r - k^2)}} , & k^2 < r < 1 \end{cases} \quad (21)$$

where  $k_s$  is the ratio of the Rayleigh wave speed  $c_s$  to the dilational wave speed  $c_1$ .

Recalling from eq.(18) that  $g(r)$  is dependent on  $\phi(r)$ , it is seen that eq.(18) does not yet solve explicitly for  $\phi(r)$ . Thus, the next step is to consider the integral on the right hand side of eq.(18). Denote this integral by  $J$ .

$$J = \int_{c/c_1}^1 \frac{(s - c^2/c_1^2) \exp[\omega(s)] b(s) q(s)}{(s-r) \sqrt{a^2(s) + b^2(s)}} ds \quad (22)$$

and rewrite  $g(r)$  in the form

$$g(r) = B + A/[r - (\dot{a}^2/c_1^2)] \quad (23)$$

where

$$A = \frac{2(P-q_0)\dot{a}(\dot{a}^2 - \dot{c}^2)}{\pi c_1^3} \quad (24)$$

$$B = \frac{2q_0 c'}{\pi c_1} + \frac{2(P-q_0)a'}{\pi c_1} + \frac{1}{\pi} \int_{c^2/c_1^2}^1 \frac{\sqrt{s - c/c_1}}{\sqrt{s}} \phi(s) ds \quad (25)$$

Then  $J$  may be rewritten as

$$J = \int_{c^2/c_1^2}^1 \frac{(s - c^2/c_1^2) \exp[\omega(s)] b(s) [q(s) - q(r) + q(r)]}{(s-r) \sqrt{a^2(s) + b^2(s)}} ds$$

or

$$J = g(r)I_1(r) + [A/((\dot{a}^2/c_1^2) - r)]I_1(\dot{a}^2/c_1^2) \quad (26)$$

where

$$I_1(r) = \int_{c_1^2/c_2^2}^1 \frac{(s - c_1^2/c_2^2) \exp[\omega(s)] b(s)}{(s-r) \sqrt{a^2(s) + b^2(s)}} ds \quad (27)$$

since

$$\frac{q(s) - q(r)}{s-r} = \frac{A}{s-r} \left[ \frac{1}{(s - a_1^2/c_1^2)} - \frac{1}{(r - a_1^2/c_1^2)} \right] = - \frac{A}{(s - a_1^2/c_1^2)(r - a_1^2/c_1^2)}$$

$I_1(r)$  and  $I_1(\dot{a}^2/c_1^2)$  may be calculated by contour integration as in Appendix 2 of Broberg the result being:

$$I_1(r) = \begin{cases} \pi(M-r) & , \quad c_1^2/c_2^2 < r < k^2 \\ \pi(M-r) + \frac{\pi 4k^3 r(r-k^2)(1-r)\sqrt{(r-k^2)(r-c_1^2/c_2^2)}}{(r-2k^2)^4 + 16k^6(1-r)(r-k^2)} & , k^2 < r < 1 \end{cases} \quad (28)$$

and

$$I_1(\dot{a}^2/c_1^2) = \pi[M - (\dot{a}^2/c_1^2)] - \pi[(\dot{a}^2/c_1^2) - k^2]T \quad (29)$$

where

$$T = \frac{(a^2/c_1^2) \sqrt{(1 - a^2/c_1^2)(c_1^2/c_2^2 - a^2/c_1^2)}}{(a_1^2/c_1^2 - 2k^2)^2 - 4k^3 \sqrt{(1 - a_1^2/c_1^2)(k^2 - a_1^2/c_1^2)}} \quad (30)$$

with M being a constant it is not necessary to evaluate.

Upon substitution of the values for  $I_1(r)$ ,  $I_1(\dot{a}^2/c_1^2)$  into  $J(r)$  and then into eq.(18) the resulting expression for  $\phi(r)$  is:

$$\phi(r) = \left\{ -\frac{B(M-r)}{r(r-k_s^2)} - \frac{A(M-r)}{r(r-k_s^2)(r-\frac{a^2}{c_1^2})} + \frac{A(M-\frac{a^2}{c_1^2})}{r(r-k_s^2)(r-\frac{a^2}{c_1^2})} + \frac{L}{r(r-k_s^2)} \right. \\ \left. - \frac{A(M-\frac{a^2}{c_1^2})}{r(r-k_s^2)(r-\frac{a^2}{c_1^2})} \right\} \cdot \begin{cases} \frac{(r-2k^2)^2 - 4k^3\sqrt{(1-r)(k^2-r)}}{\sqrt{(1-r)(r-\frac{c_1^2}{c_2^2})^3}}, & \frac{c_1^2}{c_2^2} < r < k^2 \\ \frac{(r-2k^2)^2}{\sqrt{(1-r)(r-\frac{c_1^2}{c_2^2})^3}}, & k^2 < r < 1 \end{cases} \quad (31)$$

Next, isolate coefficients of the term  $1/(r-k_s^2)$  in eq.(31), thus rewriting  $\phi(r)$  as:

$$\phi(r) = \left\{ \frac{-BM + A + TA + L + Bk_s^2}{k_s^2(r-k_s^2)} + \frac{BM - A - TA - L}{rk_s^2} \right. \\ \left. - \frac{TA}{r(r-\frac{a^2}{c_1^2})} \right\} \cdot \begin{cases} \frac{(r-2k^2)^2 - 4k^3\sqrt{(1-r)(k^2-r)}}{\sqrt{(1-r)(r-\frac{c_1^2}{c_2^2})^3}}, & \frac{c_1^2}{c_2^2} < r < k^2 \\ \frac{(r-2k^2)^2}{\sqrt{(1-r)(r-\frac{c_1^2}{c_2^2})^3}}, & k^2 < r < 1 \end{cases} \quad (32)$$

The first term on the right hand side of eq.(32) does not satisfy the displacement boundary conditions for the problem (see Broberg). Then in order that this term be zero, set

$$L = BM - Bk_s^2 - A - TA \quad (33)$$

and,  $\phi(r)$  is then,

$$\phi(r) = \left\{ B - \frac{TA}{\left(r - \frac{a^2}{c_1^2}\right)} \right\} \cdot \begin{cases} \frac{(r - 2k^2)^2 - 4k^3 \sqrt{(1-r)(k^2-r)}}{r \sqrt{(1-r)(r - \frac{c^2}{c_1^2})^3}} & , \frac{c^2}{c_1^2} < r < k^2 \\ \frac{(r - 2k^2)^2}{r \sqrt{(1-r)(r - \frac{c^2}{c_1^2})^3}} & , k^2 < r < 1 \end{cases} \quad (34)$$

In order to evaluate the unknown constant B substitute eq.(34) into eq.(25). The result of this substitution is

$$BZ_1 = (2q_0 \dot{c}/c_1) - [TA/(\dot{a}^2/c_1^2)]Z_2 \quad (35)$$

where,

$$\begin{aligned} Z_1 = & \frac{2[(c^2/c_1^2) - 4k^2(c^2/c_1^2) + 4k^4]c_1}{c(1 - c^2/c_1^2)} K(\sqrt{1 - c^2/c_1^2}) \\ & - \frac{2[(c/c_1)^4 - 4k^2(1+k^2)(c^2/c_1^2) + 8k^4]c_1^3}{c^3(1 - c^2/c_1^2)} E(\sqrt{1 - c^2/c_1^2}) \\ & - \frac{8k^2c}{c} K(\sqrt{1 - c^2/c_1^2}k^2) + \frac{16k^4c^3}{c^3} E(\sqrt{1 - c^2/c_1^2}k^2) \end{aligned} \quad (36)$$

and

$$\begin{aligned} Z_2 = & - \frac{2a^2(c^2/c_1^2 - 2k^2)^2}{c_1 c (c^2/c_1^2 - a^2/c_1^2)(1 - c^2/c_1^2)} K(\sqrt{1 - c^2/c_1^2}) \\ & + \frac{2(a^2/c_1^2)(c^2/c_1^2 - 2k^2)^2 - 8k^4(c^2/c_1^2 - a^2/c_1^2)(1 - c^2/c_1^2)}{(c^3/c_1^3)(c^2/c_1^2 - a^2/c_1^2)(1 - c^2/c_1^2)} E(\sqrt{1 - c^2/c_1^2}) \\ & + \frac{8k^2(a^2/c_1^2)}{(c/c_1)(c^2/c_1^2 - a^2/c_1^2)} K(\sqrt{1 - c^2/c_1^2}k^2) + \frac{8k^4(c^2/c_1^2 - 2a^2/c_1^2)}{(c^3/c_1^3)(c^2/c_1^2 - a^2/c_1^2)} E(\sqrt{1 - c^2/c_1^2}k^2) + \end{aligned}$$

$$\begin{aligned}
& + \frac{2(c/c_1)(a^2/c_1^2 - 2k^2)^2}{(1 - a^2/c_1^2)(c^2/c_1^2 - a^2/c_1^2)} \Pi\left(\frac{(1 - c^2/c_1^2)}{(1 - a^2/c_1^2)}, \sqrt{1 - c^2/c_1^2}\right) \\
& - \frac{8k^2 c'}{c_1(c^2/c_1^2 - a^2/c_1^2)} \Pi\left(\frac{(1 - c^2/c_1^2 k^2)}{(1 - a^2/c_1^2 k^2)}, \sqrt{1 - c^2/c_1^2 k^2}\right)
\end{aligned}
\tag{37}$$

where  $K$  and  $E$  are complete elliptic integrals of the first and second kind and where  $\Pi$  is a complete elliptic integral of the third kind. (see Byrd and Friedman, Handbook of Elliptic Integrals).

STRESS ON SURFACE  $z = 0$

The expression for stress over the interval  $ct < |x| < vt$  on the surface  $z = 0$  may be found by integrating eq.(6) over the range of eq.(7) and recalling <sup>the</sup> transformation of variables in eq.(11). Thus,

$$\sigma_z(x) = - \int_c^{c_1} q'(\omega) d\omega = - \frac{1}{2} \int_{x^2/c_1^2 t^2}^1 \frac{\phi(s)}{\sqrt{s}} ds \tag{38}$$

which upon substitution of eq.(34) reads,

$$\begin{aligned}
\sigma_z(x) = & - \frac{B}{2} \left\{ \int_{x^2/c_1^2 t^2}^1 \frac{(s - 2k^2)^2 ds}{s \sqrt{s(1-s)(s - c^2/c_1^2)^3}} - 4k^3 \int_{x^2/c_1^2 t^2}^{k^2} \frac{\sqrt{k^2 - s} ds}{s \sqrt{s(s - c^2/c_1^2)^3}} \right\} \\
& + \frac{TA}{2} \left\{ \int_{x^2/c_1^2 t^2}^1 \frac{(s - 2k^2)^2 ds}{s(s - a^2/c_1^2) \sqrt{s(1-s)(s - c^2/c_1^2)^3}} - 4k^3 \int_{x^2/c_1^2 t^2}^{k^2} \frac{\sqrt{k^2 - s} ds}{s(s - a^2/c_1^2) \sqrt{s(s - c^2/c_1^2)^3}} \right\}
\end{aligned}
\tag{39}$$

$$ct < |x| < c_1 t$$



The objective next is to make the stress singularity at  $|x| = \dot{c}t$  equal to zero. This is accomplished if

$$B = AT/[(\dot{c}/c_1)^2 - (\dot{a}/c_1)^2] \quad (40)$$

with the stress expression becoming,

$$\sigma_z(x) = -\frac{ATc_1^2}{2(c_1^2 - a^2)} \left\{ \int_{\frac{x^2}{c_1^2 + t^2}}^1 \frac{(s - 2k^2)^2 ds}{s(s - \frac{a^2}{c_1^2})/s(1-s)(s - \frac{c_1^2}{c_1^2})^3} - 4k^3 \int_{\frac{x^2}{c_1^2 + t^2}}^{k^2} \frac{\sqrt{k^2 - s} ds}{s(s - \frac{a^2}{c_1^2})/s(s - \frac{c_1^2}{c_1^2})} \right\}, \quad c_1 t < |x| < c_1 t \quad (41)$$

where it can be seen by inspection of the integrands in eq.(41) that the integrals remain bounded when  $|x| = \dot{c}t$ .

## DISPLACEMENTS

The region of non-zero displacement on the surface  $z = 0$  is  $|x| < \dot{c}t$ . For this range, eq.(10) reduces to,

$$\frac{\partial^2 u_z}{\partial t^2} = -\frac{\text{Re}[F(\frac{c_1^2 + t^2}{x^2})]}{2\pi k^4 e c_1^2 |x|} \left\{ \frac{q_0 c_1 c_1}{1 - \frac{c_1^2 + t^2}{x^2}} + \frac{(P - q_0) a c_1}{1 - \frac{a^2 + t^2}{x^2}} + \int_{c_1}^{c_1} \left[ \frac{r c_1}{1 - \frac{r^2 + t^2}{x^2}} - \frac{e c_1}{1 - \frac{e^2 + t^2}{x^2}} \right] q'(r) dr \right\}, \quad z=0, \quad 0 < |x| < c_1 t$$

The change of variables in equation (11) will again be used. Thus, eq.(42) is rewritten in the form,

$$\frac{\partial^2 u_z}{\partial t^2} = \frac{-\text{Re}[F(1/r)]}{2\pi k^4 e c_1^2 |x|} \left\{ \frac{q_0 c_1 r}{(r - c^2/c_1^2)} + \frac{(P - q_0) a' c_1 r}{(r - a'^2/c_1^2)} \right. \\ \left. + \frac{c_1^2}{2} \int_{c^2/c_1^2}^1 \left[ \frac{r}{r-s} - \frac{(c/c_1) r}{\sqrt{s} (r - a'^2/c_1^2)} \right] \phi(s) ds \right\}, \quad 0 < |x| < ct \\ \text{or, equivalently, } 0 < r < c^2/c_1^2 \quad (43)$$

Eq.(43) may be rewritten as

$$\frac{\partial^2 u_z}{\partial t^2} = \frac{-r \text{Re}[F(1/r)]}{2\pi k^4 e |x| (r - c^2/c_1^2)} \left\{ \frac{q_0 c}{c_1} + \frac{(r - c^2/c_1^2) (P - q_0) a'}{(r - a'^2/c_1^2) c_1} \right. \\ \left. - \frac{1}{2} \int_{c^2/c_1^2}^1 \frac{(s - c^2/c_1^2)}{s - r} \phi(s) ds + \frac{1}{2} \int_{c^2/c_1^2}^1 \frac{\sqrt{s} - c/c_1}{\sqrt{s}} \phi(s) ds \right\}, \\ 0 < r < c^2/c_1^2 \quad (44)$$

Further simplification is allowed through eq.(25) so that the preceding expression becomes

$$\frac{\partial^2 u_z}{\partial t^2} = \frac{-r \text{Re}[F(1/r)]}{2\pi k^4 e |x| (r - c^2/c_1^2)} \left\{ \frac{\pi B}{2} + \frac{(a'^2 - c^2) (P - q_0) a'}{c_1^3 (r - a'^2/c_1^2)} \right. \\ \left. - \frac{1}{2} \int_{c^2/c_1^2}^1 \frac{(s - c^2/c_1^2)}{(s - r)} \phi(s) ds \right\}, \quad 0 < r < c^2/c_1^2 \quad (45)$$

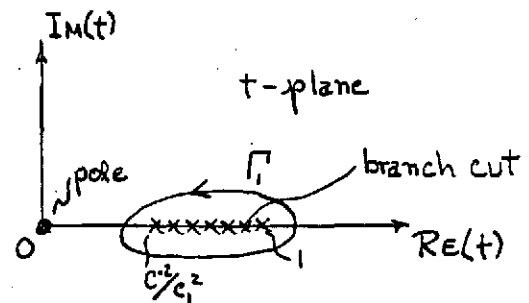
It remains to evaluate the integral on the right hand side of eq.(45). Making use of eq.(34)

$$\begin{aligned} \int_{\frac{c^2}{c_1^2}}^1 \frac{(s - \frac{c^2}{c_1^2})}{(s-r)} \phi(s) ds = & B \left\{ \int_{\frac{c^2}{c_1^2}}^1 \frac{(s-2k^2)^2 ds}{s(s-r)/(1-s)(s-\frac{c^2}{c_1^2})} \right. \\ & - 4k^3 \int_{\frac{c^2}{c_1^2}}^k \frac{\sqrt{\frac{(k^2-s)}{(s-\frac{c^2}{c_1^2})}}}{s(s-r)} ds \left. \right\} - TA \left\{ \int_{\frac{c^2}{c_1^2}}^1 \frac{(s^2-2k^2)^2 ds}{(s-\frac{c^2}{c_1^2})s(s-r)/(1-s)(s-\frac{c^2}{c_1^2})} \right. \\ & \left. - 4k^3 \int_{\frac{c^2}{c_1^2}}^k \frac{\sqrt{\frac{(k^2-s)}{(s-\frac{c^2}{c_1^2})}}}{s(s-\frac{c^2}{c_1^2})(s-r)} ds \right\} \quad (46) \end{aligned}$$

The four integrals in the preceding expression will be evaluated by contour integration (see for example p. 276 of Muskhelishvili)\* Note that all the following are evaluated for the range  $0 < r < (\dot{c}^2/c_1^2)$ .

$$I_1 = \int_{\frac{c^2}{c_1^2}}^1 \frac{(s-2k^2)^2 ds}{s(s-r)/(1-s)(s-\frac{c^2}{c_1^2})} \quad (47)$$

The real integral  $I_1$  may be written as the limit of a contour integral in the  $t$ -plane.



$$\begin{aligned} 2I_1 &= \lim_{\Gamma} \frac{1}{i} \int_{\Gamma} \frac{(t-2k^2)^2 dt}{t(t-r)/(t-1)(t-\frac{c^2}{c_1^2})} \\ &= -2\pi \left\{ f^I(r) - G_1^I(r) - G_\infty^I(r) \right\}, \quad 0 < r < \frac{c^2}{c_1^2} \end{aligned}$$

where

$$f^I(r) = - \frac{(r-2k^2)^2}{r\sqrt{(1-r)[(\dot{c}^2/c_1^2)-r]}}$$

\* Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity

$$G_1^I(r) = -\frac{4k^4 c_1}{cr}$$

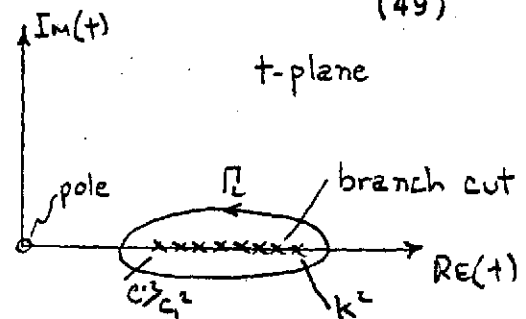
$$G_\infty^I(r) = 1$$

Then,

$$I_1 = \pi \left\{ \frac{(r-2k^2)^2}{r\sqrt{(1-r)(c_2^2/c_1^2-r)}} - \frac{4k^2 c_1}{c \cdot r} + 1 \right\} \quad (48)$$

$$I_2 = \int_{c_2^2/c_1^2}^{k^2} \sqrt{\frac{(k^2-s)}{(s-c_2^2/c_1^2)}} \frac{ds}{s(s-r)}$$

(49)



$$2I_2 = \lim_{\frac{1}{i}} \int_{\Gamma_2} \frac{(k^2-t)dt}{t(t-r)\sqrt{(t-k^2)(t-c_2^2/c_1^2)}}$$

$$= -2\pi \left\{ f^{II}(r) - G_1^{II}(r) - G_\infty^{II}(r) \right\}$$

where,

$$f^{II}(r) = \frac{r-k^2}{r\sqrt{(k^2-r)(c_2^2/c_1^2-r)}}$$

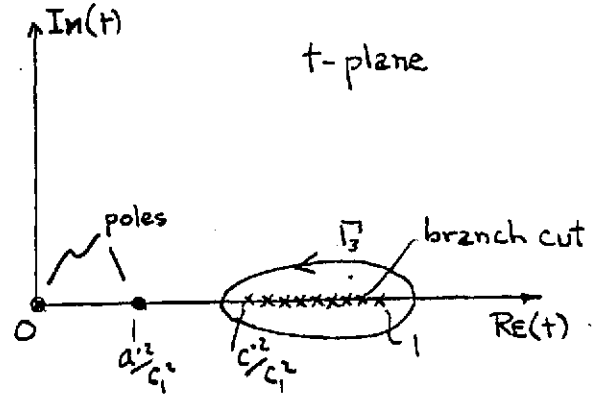
$$G_1^{II}(r) = -\frac{kc_1}{cr}$$

$$G_\infty^{II}(r) = 0$$

Then,

$$I_2 = \sqrt{\frac{(k^2 - r)}{[(\dot{c}^2/c_1^2) - r]}} \frac{1}{r} - \frac{kc_1}{\dot{c}r} \quad (50)$$

$$I_3 = \int \frac{(s^2 - 2k^2)^2 ds}{(\dot{c}^2/c_1^2) (s - a^2/c_1^2) s(s-r) \sqrt{(1-s)(s - c^2/c_1^2)}} \quad (51)$$



$$= \lim_{\epsilon \rightarrow 0} \frac{1}{i} \int_{\Gamma} \frac{(t - 2k^2)^2 dt}{(t - a^2/c_1^2)(t - r) t \sqrt{(t-1)(t - c^2/c_1^2)}}$$

$$= -2\pi \left\{ f^{\text{III}}(r) - G_1^{\text{III}}(r) - G_2^{\text{III}}(r) - G_\infty^{\text{III}}(r) \right\}$$

where

$$f^{\text{III}}(r) = - \frac{(r - 2k^2)^2}{r(r - a^2/c_1^2) \sqrt{(1-r)(c^2/c_1^2 - r)}}$$

$$G_1^{\text{III}}(r) = 4k^4 c_1^3 / a^2 \dot{c} r$$

$$G_2^{\text{III}}(r) = \frac{(a^2/c_1^2 - 2k^2)^2}{(a^2/c_1^2) (r - a^2/c_1^2) \sqrt{(c^2/c_1^2 - a^2/c_1^2)(r - a^2/c_1^2)}}$$

$$G_\infty^{\text{III}}(r) = 0$$

Then,

$$I_3 = \pi \left\{ \frac{(r - 2k^2)^2}{r(r - a^2/c_1^2) \sqrt{(1-r)(c^2/c_1^2 - r)}} + \frac{4k^4 c_1^3}{a^2 \dot{c} r} - \frac{(a^2/c_1^2 - 2k^2)^2 c_1^2}{(r - a^2/c_1^2) a^2 \sqrt{(1 - a^2/c_1^2)(c^2/c_1^2 - a^2/c_1^2)}} \right\} \quad (52)$$

$$I_4 = \int_{\frac{c_1^2}{c_1^2}}^{k^2} \sqrt{\frac{(k^2-s)}{(s-\frac{a^2}{c_1^2})}} \frac{ds}{s(s-r)(s-\frac{a^2}{c_1^2})} \quad (53)$$

$$2I_4 = \lim_{\frac{1}{2}} \int_{\Gamma_4} \frac{(k^2-t) dt}{t(t-r)(t-\frac{a^2}{c_1^2}) \sqrt{(t-k^2)(t-\frac{c_1^2}{c_1^2})}}$$

$$= -2\pi \left\{ f^{\text{IV}}(r) - G_1^{\text{IV}}(r) - G_2^{\text{IV}}(r) - G_\infty^{\text{IV}}(r) \right\}$$

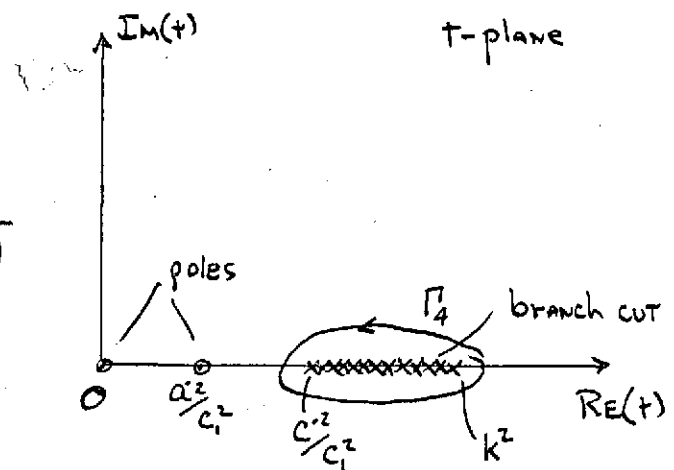
where,

$$f^{\text{IV}}(r) = \frac{(r-k^2)}{r(r-\frac{a^2}{c_1^2}) \sqrt{(k^2-r)(\frac{c_1^2}{c_1^2}-r)}}$$

$$G_1^{\text{IV}}(r) = \frac{kc_1^3}{a^2 c_1 r}$$

$$G_2^{\text{IV}}(r) = - \frac{(k^2 - \frac{a^2}{c_1^2})}{(\frac{a^2}{c_1^2})(r - \frac{a^2}{c_1^2}) \sqrt{(k^2 - \frac{a^2}{c_1^2})(\frac{c_1^2}{c_1^2} - \frac{a^2}{c_1^2})}}$$

$$G_\infty^{\text{IV}}(r) = 0$$



Hence,

$$I_4 = \pi \left\{ \frac{\sqrt{(k^2-r)}}{\sqrt{(\frac{c_1^2}{c_1^2}-r)}} \frac{1}{r(r-\frac{a^2}{c_1^2})} + \frac{kc_1^3}{a^2 c_1 r} - \frac{c_1^2}{a^2} \frac{\sqrt{(k^2-\frac{a^2}{c_1^2})}}{(\frac{c_1^2}{c_1^2}-\frac{a^2}{c_1^2})} \frac{1}{(r-\frac{a^2}{c_1^2})} \right\} \quad (54)$$

Using the preceding expressions for  $I_1$ ,  $I_2$ ,  $I_3$ , and  $I_4$  eq.(46) reduces to,

$$\int_{c_1^2/c_2^2}^1 \frac{(s - c_1^2/c_2^2)}{(s-r)} \phi(s) ds = \pi B \left\{ \frac{(r-2k^2)^2 - 4k^3\sqrt{(1-r)(k^2-r)}}{r\sqrt{(1-r)(c_1^2/c_2^2-r)}} + 1 \right\} \\ - \pi TA \left\{ \frac{(r-2k^2)^2 - 4k^3\sqrt{(1-r)(k^2-r)}}{r(r - a_1^2/c_1^2)\sqrt{(1-r)(c_1^2/c_2^2-r)}} \right. \\ \left. - \frac{(a_1^2/c_1^2 - 2k^2)^2 - 4k^3\sqrt{(1-a_1^2/c_1^2)(k^2 - a_1^2/c_1^2)}}{(a_1^2/c_1^2)(r - a_1^2/c_1^2)\sqrt{(1-a_1^2/c_1^2)(c_1^2/c_2^2 - a_1^2/c_1^2)}} \right\}. \quad (56)$$

and

$$\frac{\partial^2 v_z}{\partial t^2} = \frac{r \operatorname{Re}[F(1/r)]}{2\pi e k^4 |x| (c_1^2/c_2^2 - r)} \left[ \frac{(a_1^2/c_1^2 - c_1^2/c_2^2)(P-q_0)(a_1/c_1)}{(r - a_1^2/c_1^2)} \right. \\ - \frac{\pi B}{2} \frac{(r-2k^2)^2 - 4k^3\sqrt{(1-r)(k^2-r)}}{r\sqrt{(1-r)(c_1^2/c_2^2-r)}} \\ + \frac{\pi TA}{2} \left\{ \frac{(r-2k^2)^2 - 4k^3\sqrt{(1-r)(k^2-r)}}{r(r - a_1^2/c_1^2)\sqrt{(1-r)(c_1^2/c_2^2-r)}} \right. \\ \left. - \frac{(a_1^2/c_1^2 - 2k^2)^2 - 4k^3\sqrt{(1-a_1^2/c_1^2)(k^2 - a_1^2/c_1^2)}}{(a_1^2/c_1^2)(r - a_1^2/c_1^2)\sqrt{(1-a_1^2/c_1^2)(c_1^2/c_2^2 - a_1^2/c_1^2)}} \right\} \Big], 0 < r < \frac{c_1^2}{c_2^2} \quad (57)$$

But, recalling eq.(4),

$$\operatorname{Re}[F(1/r)] = \frac{4k^4\sqrt{(1-r)} r^{3/2}}{(r-2k^2)^2 - 4k^3\sqrt{(k^2-r)(1-r)}} , 0 < r < \frac{c_1^2}{c_2^2}, \quad (58)$$

and making use of eqs.(24) and (30), eq.(57) reduces to

$$\frac{\partial^2 u_z}{\partial t^2} = \frac{-r}{e|x|(\frac{c^2}{c_1^2}-r)} \left\{ \frac{Br^{1/2}}{\sqrt{\frac{c^2}{c_1^2}-r}} - \frac{TA r^k}{(r-\frac{a^2}{c_1^2})\sqrt{\frac{c^2}{c_1^2}-r}} \right\} \quad r < \frac{c^2}{c_1^2} \quad (59)$$

For the first integration note the following

$$\frac{\partial}{\partial t} = \frac{\partial r}{\partial t} \frac{\partial}{\partial r} = -2 \frac{x^2}{c_1^2 t^3} \frac{\partial}{\partial r} = -2 \frac{c_1 r^{3/2}}{|x|} \frac{\partial}{\partial r}$$

Hence,

$$\frac{\partial}{\partial r} \left( \frac{\partial u_z}{\partial t} \right) = \frac{1}{2ec_1} \left\{ \frac{B}{(\frac{c^2}{c_1^2}-r)^{3/2}} - \frac{TA}{(r-\frac{a^2}{c_1^2})(\frac{c^2}{c_1^2}-r)^{3/2}} \right\} \quad (60)$$

and

$$\frac{\partial u_z}{\partial t} = \frac{1}{2ec_1} \left\{ \int \frac{B dr}{(\frac{c^2}{c_1^2}-r)^{3/2}} - \frac{TA}{(\frac{c^2}{c_1^2}-\frac{a^2}{c_1^2})} \int \frac{dr}{(r-\frac{a^2}{c_1^2})\sqrt{\frac{c^2}{c_1^2}-r}} - \frac{TA}{(\frac{c^2}{c_1^2}-\frac{a^2}{c_1^2})} \int \frac{dr}{(\frac{c^2}{c_1^2}-r)^{3/2}} \right\} \quad (61)$$

Making use of eq. 212.9a of Integratafel,

$$\frac{\partial u_z}{\partial t} = \frac{1}{2ec_1} \left\{ 2 \left[ B - \frac{TA}{(\frac{c^2}{c_1^2}-\frac{a^2}{c_1^2})} \right] \frac{1}{(\frac{c^2}{c_1^2}-r)^{1/2}} - \frac{TA}{(\frac{c^2}{c_1^2}-\frac{a^2}{c_1^2})^{3/2}} \ln \left| \frac{\sqrt{\frac{c^2}{c_1^2}-r} - \sqrt{\frac{c^2}{c_1^2}-\frac{a^2}{c_1^2}}}{\sqrt{\frac{c^2}{c_1^2}-r} + \sqrt{\frac{c^2}{c_1^2}-\frac{a^2}{c_1^2}}} \right| \right\} \quad (62)$$

Recalling the finiteness condition as expressed by eq.(40),

eq.(62) reduces to

$$\frac{\partial u_z}{\partial t} = \frac{-TA}{2ec_1(\frac{c^2}{c_1^2}-\frac{a^2}{c_1^2})^{3/2}} \ln \left| \frac{\sqrt{\frac{c^2}{c_1^2}-r} - \sqrt{\frac{c^2}{c_1^2}-\frac{a^2}{c_1^2}}}{\sqrt{\frac{c^2}{c_1^2}-r} + \sqrt{\frac{c^2}{c_1^2}-\frac{a^2}{c_1^2}}} \right|, \quad 0 < r < \frac{c^2}{c_1^2} \quad (63)$$



Again writing the derivative in terms of  $r$ , a second integration with respect to  $r$  leads to the result,

$$\begin{aligned}
 u_z = \frac{-|x|TA}{2ec_1^2(c_1^2 - a^2/c_1^2)^{3/2}} & \left\{ \frac{1}{r^{1/2}} \ln \left| \frac{\sqrt{c_1^2 - a^2/c_1^2} - \sqrt{c_1^2 - r}}{\sqrt{c_1^2 - a^2/c_1^2} + \sqrt{c_1^2 - r}} \right| \right. \\
 & \left. + \frac{c_1}{a} \ln \left| \frac{r^{1/2} \sqrt{c_1^2 - a^2/c_1^2} + a/c_1 \sqrt{c_1^2 - r}}{r^{1/2} \sqrt{c_1^2 - a^2/c_1^2} - a/c_1 \sqrt{c_1^2 - r}} \right| \right\} \quad 0 \leq r < \frac{c_1^2}{c_1^2}
 \end{aligned} \quad (64)$$

In terms of  $x$  and  $t$ , the result of eq.(64) may be written

$$\begin{aligned}
 u_z(x,t) = \frac{-TA c_1^2}{2e(c^2 - a^2)^{3/2} a} & \left\{ at \ln \left| \frac{t \sqrt{c^2 - a^2} - \sqrt{c^2 t^2 - x^2}}{t \sqrt{c^2 - a^2} + \sqrt{c^2 t^2 - x^2}} \right| \right. \\
 & \left. + x \ln \left| \frac{x \sqrt{c^2 - a^2} + a \sqrt{c^2 t^2 - x^2}}{x \sqrt{c^2 - a^2} - a \sqrt{c^2 t^2 - x^2}} \right| \right\} \quad |x| < ct
 \end{aligned} \quad (65)$$

Half of the cracks opening displacement,  $\delta_T$ , is found by letting  $|x| \rightarrow at$ . The result is

$$\delta_T = 2u_z(at, t) = \frac{2(at)TA c_1^2}{e a (c^2 - a^2)} \ln \left( \frac{c}{a} \right) \quad (66)$$

It should be noted that the expressions for displacement make use of the finiteness condition, eq.(40). Then, by eqs. (35), (36), (37) and (40) the following relationship between  $\dot{a}$ ,  $\dot{c}$ ,  $P$  and  $Y$  is obtained.

$$\begin{aligned}
1 - P_Y = \frac{AT}{Y} & \left\{ - \frac{1}{\left( \frac{c^2}{a^2} - \frac{a^2}{c_1^2} \right)} K(\sqrt{1 - \frac{c^2}{c_1^2}}) \right. \\
& + \frac{4k^4 c_1^4}{a^2 c^2 \left( \frac{c^2}{c_1^2} - \frac{a^2}{c_1^2} \right)} E(\sqrt{1 - \frac{c^2}{c_1^2}}) \\
& - \frac{4k^4 c_1^4}{a^2 c^2 \left( \frac{c^2}{c_1^2} - \frac{a^2}{a_1^2} \right)} E(\sqrt{1 - \frac{c^2}{c_1^2 k^2}}) \\
& + \frac{4k^2 c_1^2}{a^2 \left( \frac{c^2}{c_1^2} - \frac{a^2}{c_1^2} \right)} \Pi \left( \frac{1 - \frac{c^2}{c_1^2 k^2}}{1 - \frac{a^2}{c_1^2 k^2}}, \sqrt{1 - \frac{c^2}{c_1^2 k^2}} \right) \\
& \left. - \frac{\left( \frac{a^2}{c_1^2} - 2k^2 \right)^2 c_1^2}{a^2 \left( 1 - \frac{a^2}{c_1^2} \right) \left( \frac{c^2}{c_1^2} - \frac{a^2}{c_1^2} \right)} \Pi \left( \frac{1 - \frac{c^2}{c_1^2}}{1 - \frac{a^2}{c_1^2}}, \sqrt{1 - \frac{c^2}{c_1^2}} \right) \right\}
\end{aligned}$$

(67)

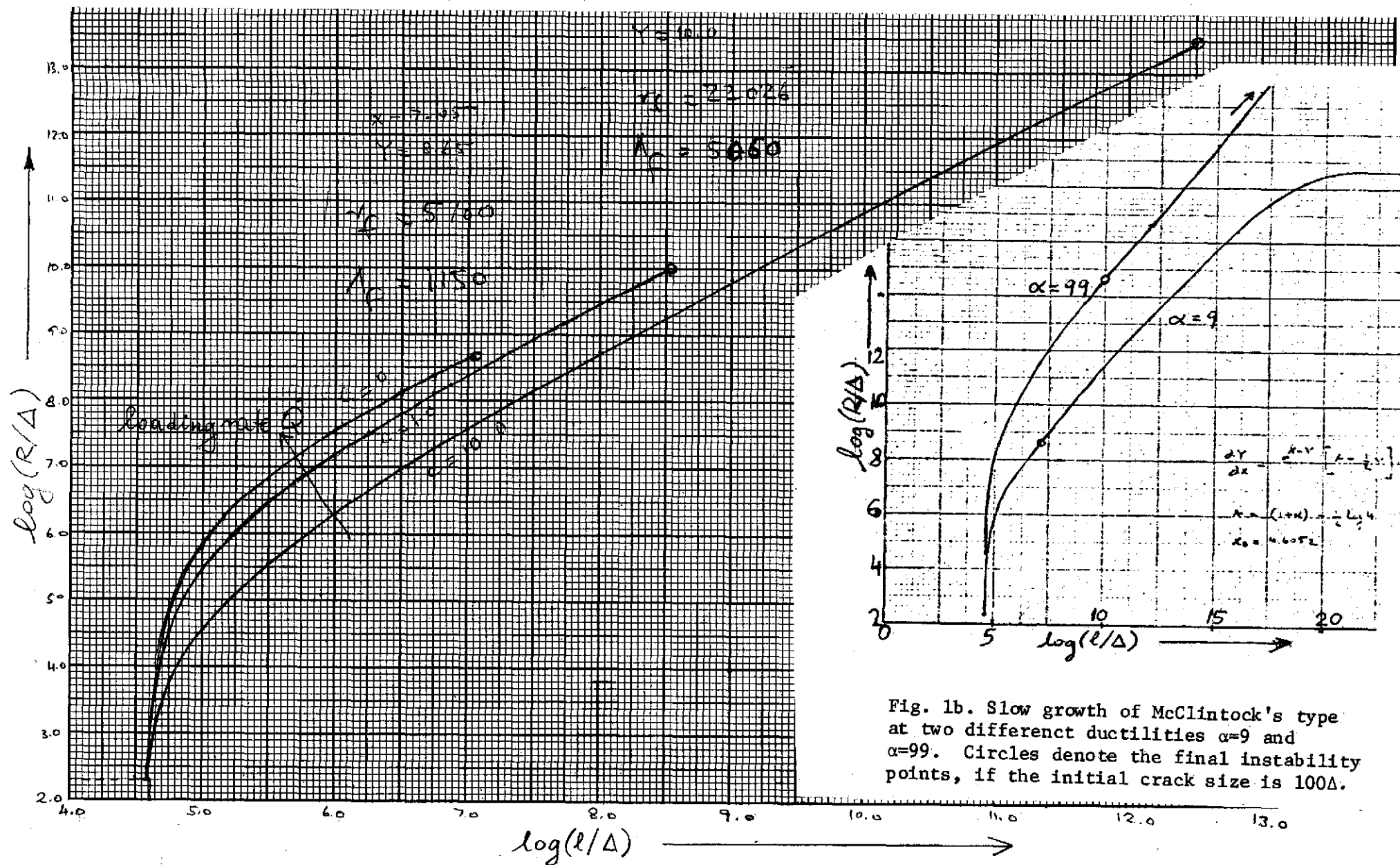


Fig. 1a. R-curves for a rate sensitive Tresca solid at various levels of the rate sensitivity/speed of loading ratio. Ultimate instability is reached when the slope of the R-curve plotted on log-log scale equals unity (marked by circles). Crack of initial length equal  $100 \times$  characteristic dimension of Neuber's domain is embedded in an infinite plate.

Fig. 1b. Slow growth of McClintock's type at two different ductilities  $\alpha=9$  and  $\alpha=99$ . Circles denote the final instability points, if the initial crack size is  $100\Delta$ .

Fig. 1c Slow growth at various loading rates. Loading parameter is plotted vs. current crack length at a chosen loading rate and material rheological sensitivity. Initial crack size is  $10R_{\infty}$ .

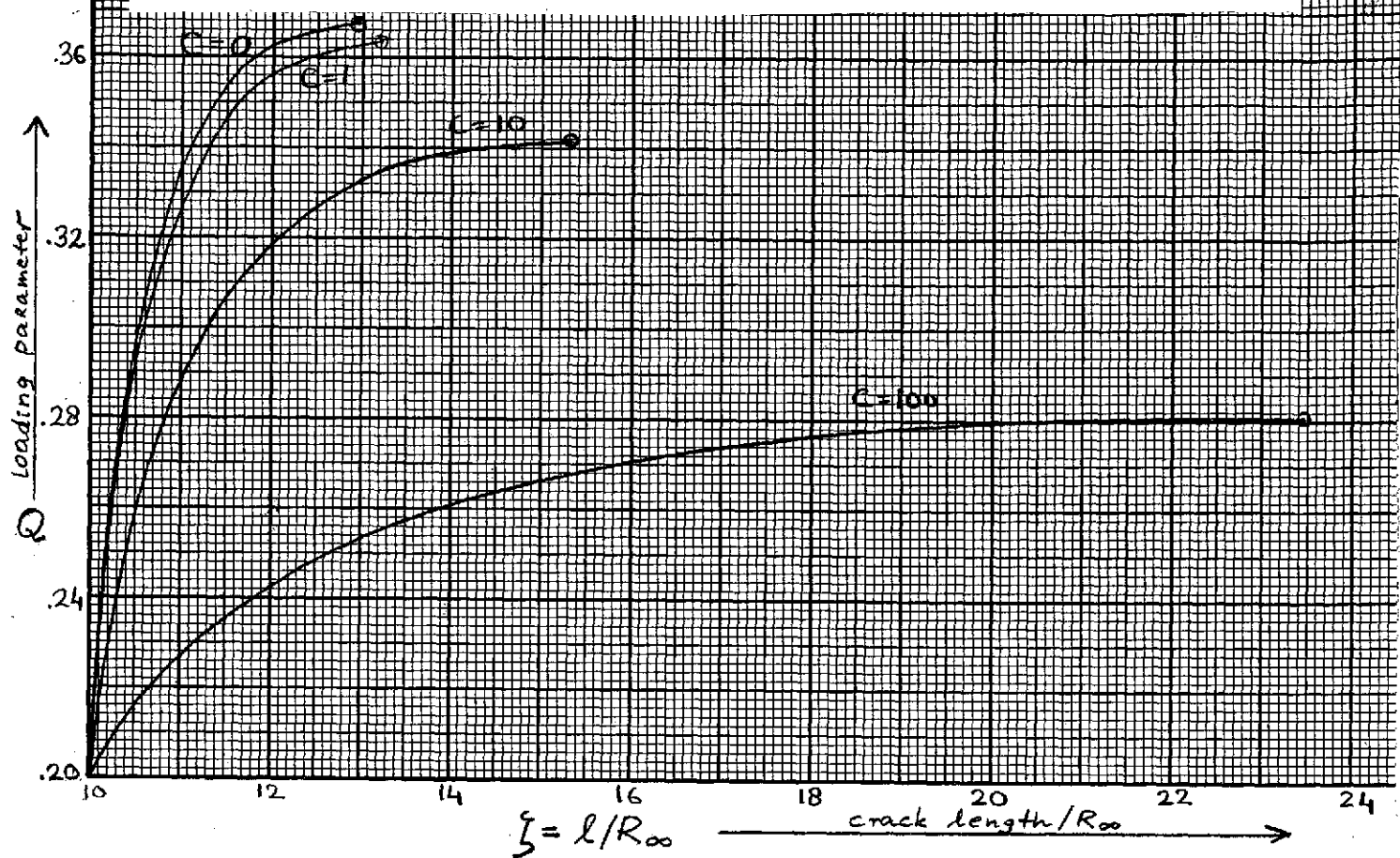
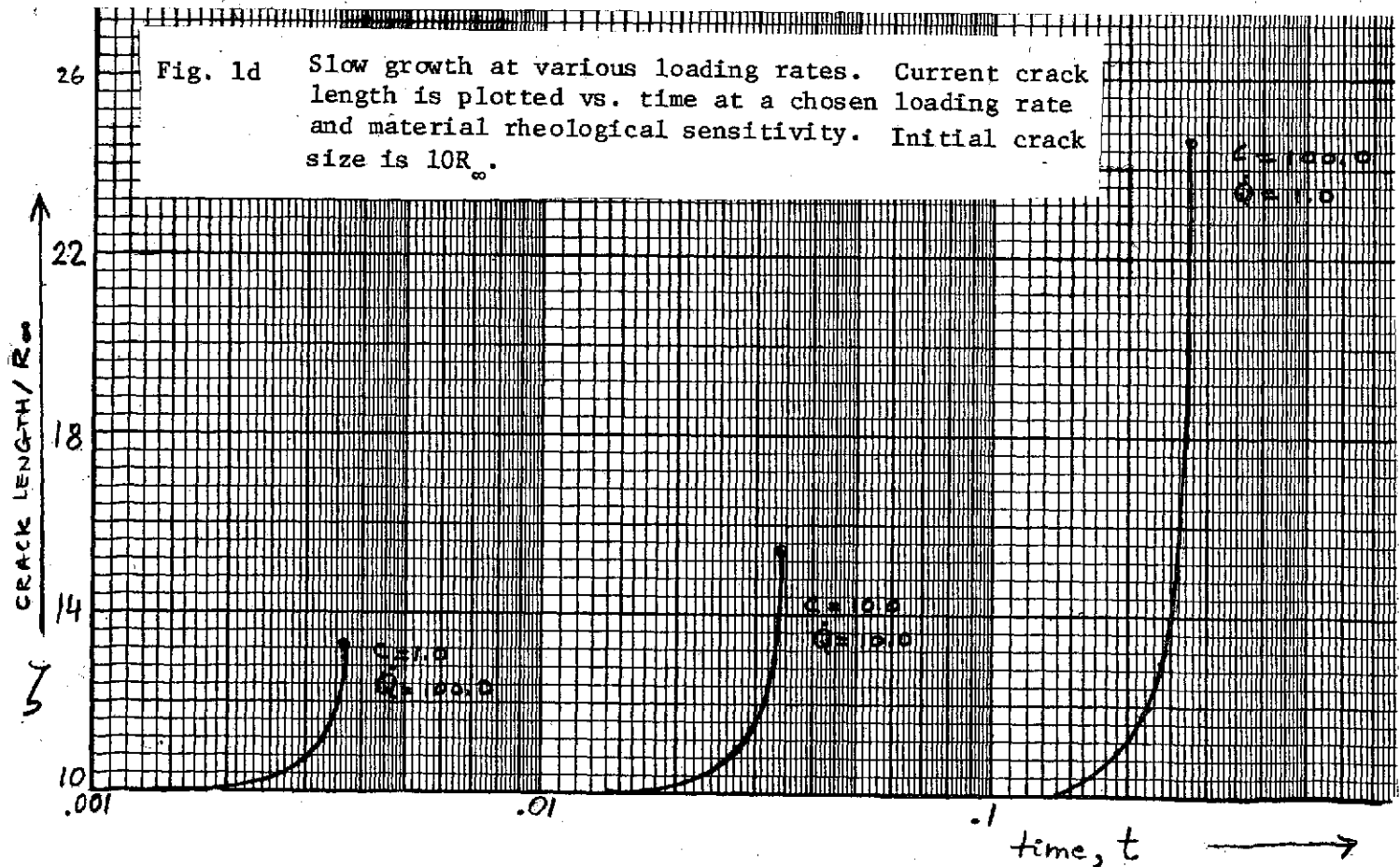


Fig. 1d Slow growth at various loading rates. Current crack length is plotted vs. time at a chosen loading rate and material rheological sensitivity. Initial crack size is  $10R_{\infty}$ .



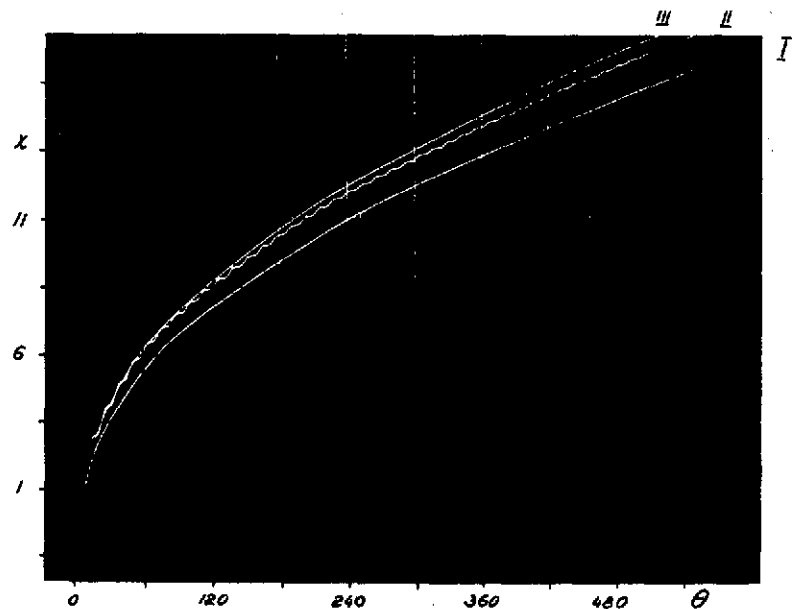
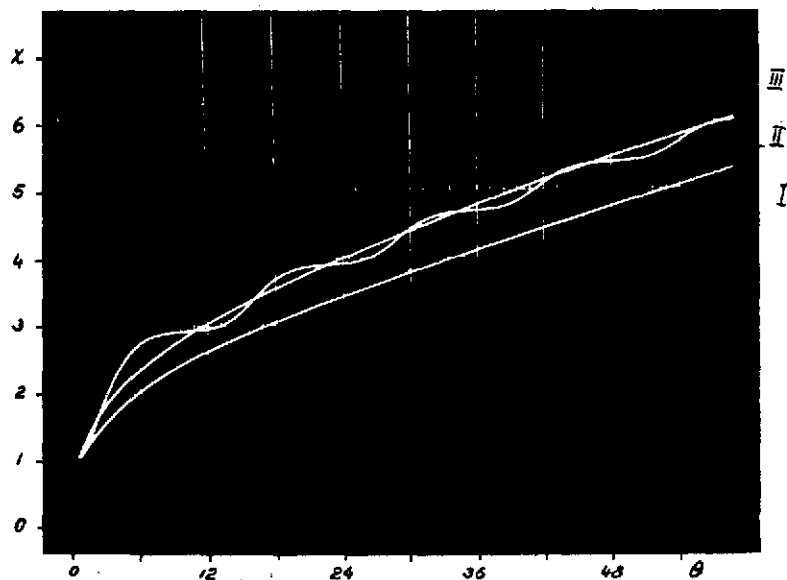


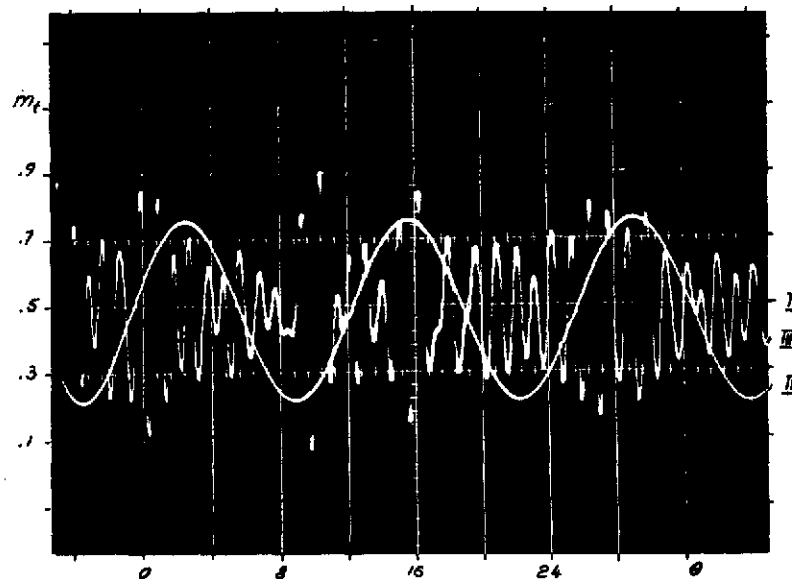
Fig. 2

(a)

Fig. 2. Subcritical growth of a crack embedded in a viscoelastic solid. Concentrated loads are applied directly to the crack surface.



(b)



(c)

Fig. 3. Subcritical growth of a crack embedded in a viscoelastic solid and driven by a remotely applied tensile stress.

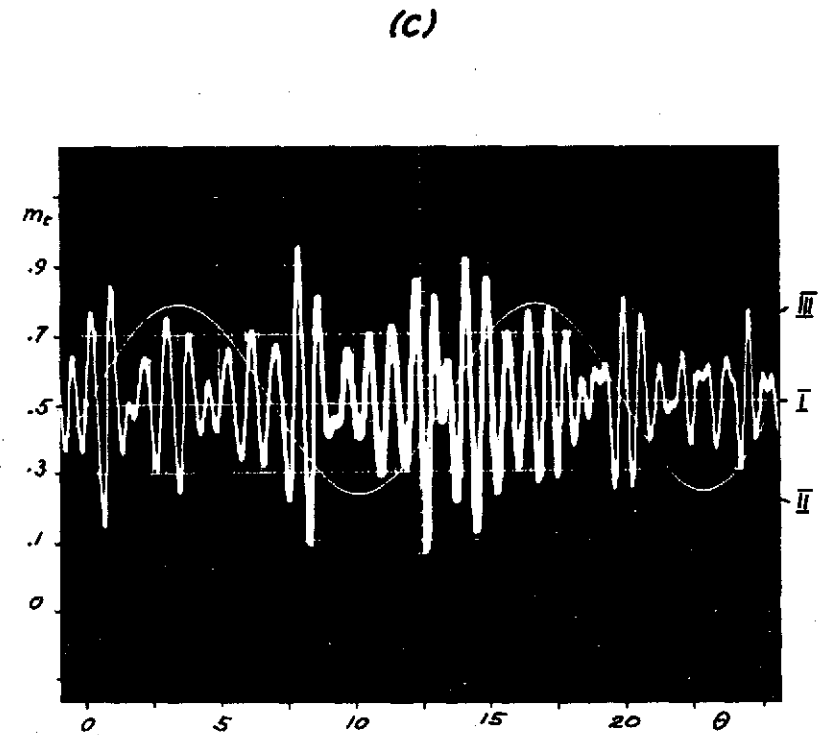
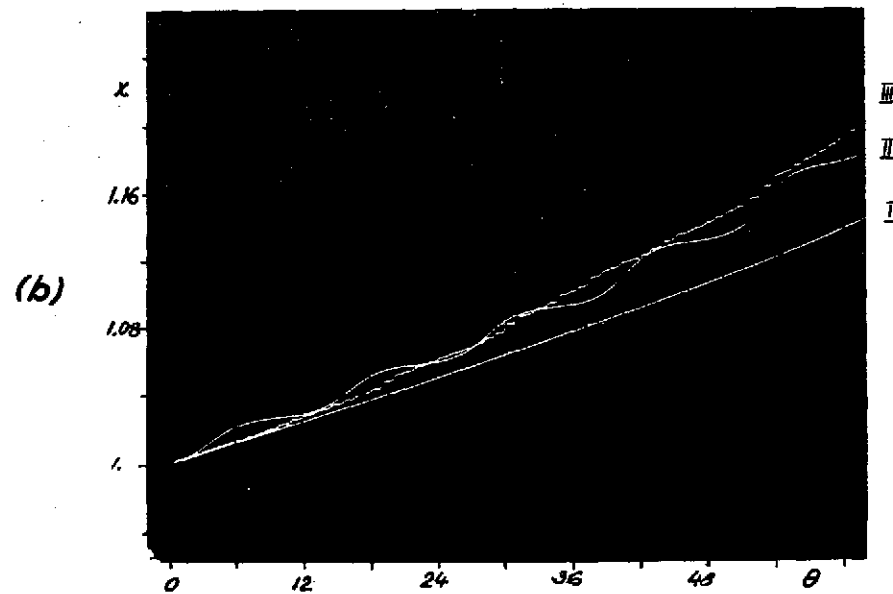
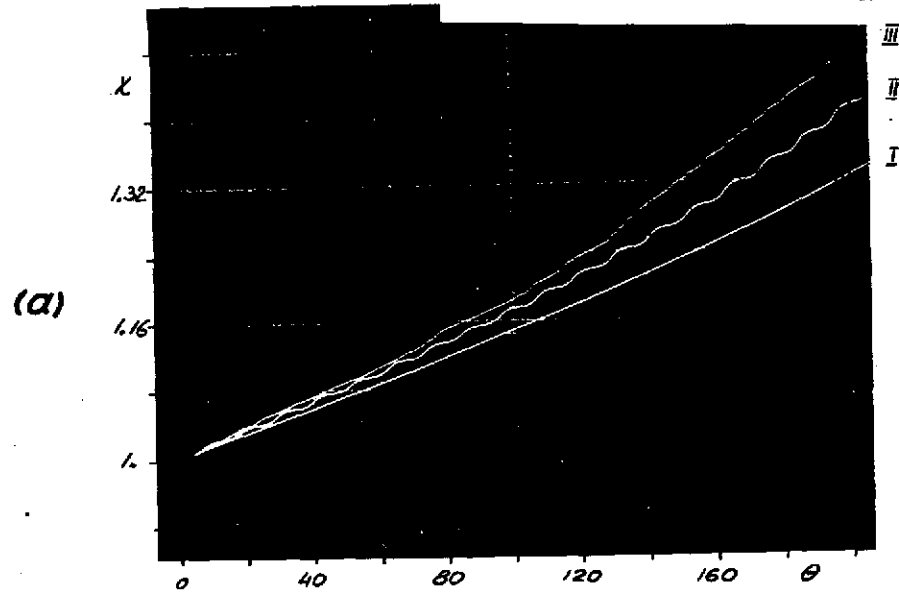
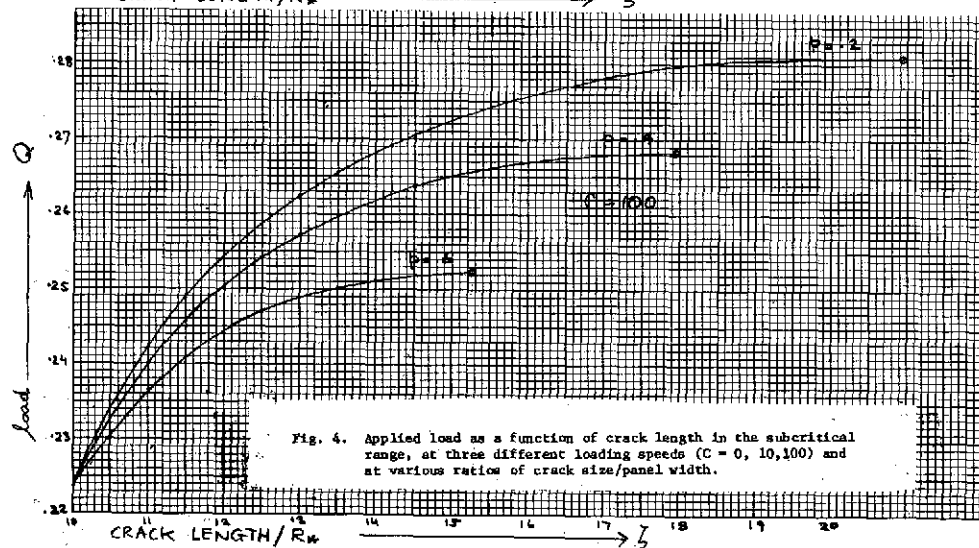
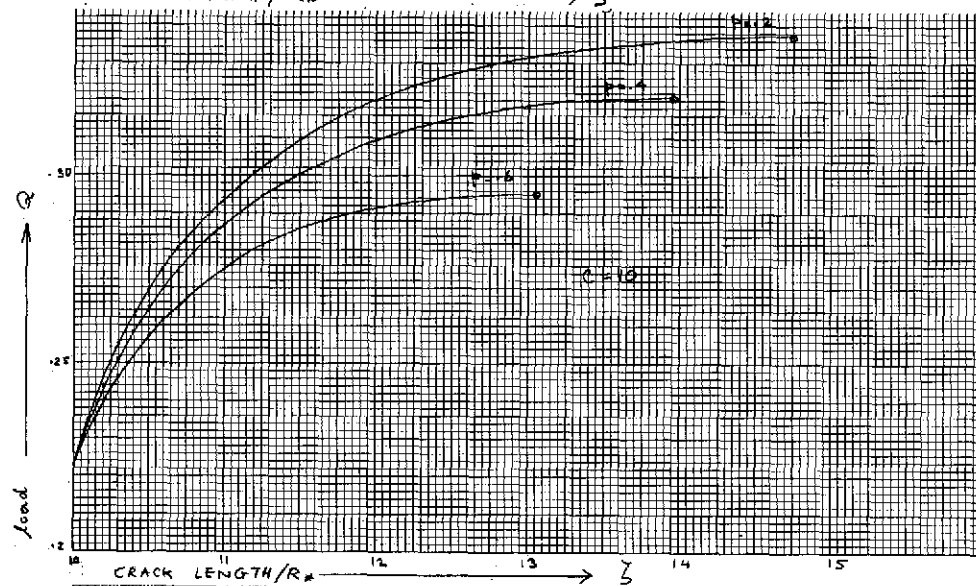
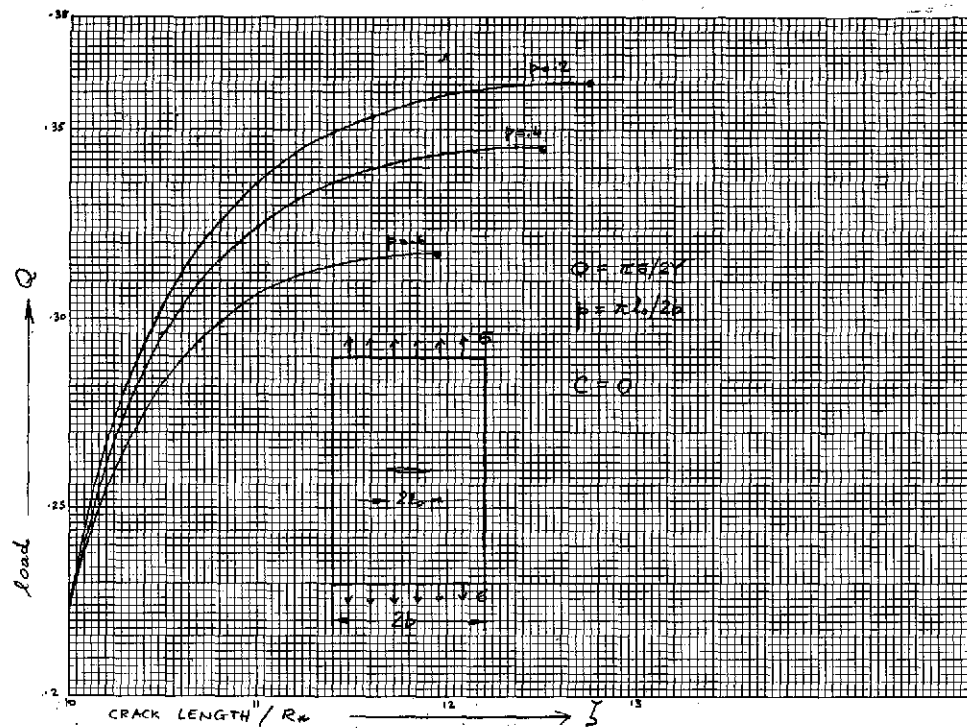


Fig. 3



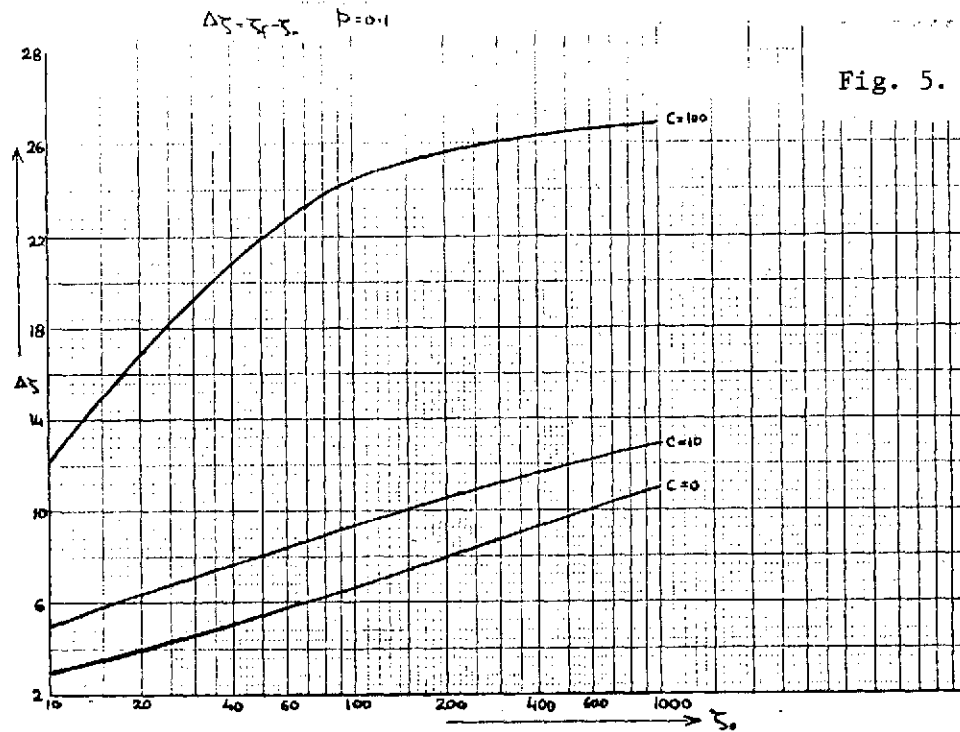


Fig. 5 Amount of slow growth  $\Delta\zeta = \zeta_f - \zeta_0$  vs. initial crack size at various rates of loadings and a constant initial crack length/panel width ratio.

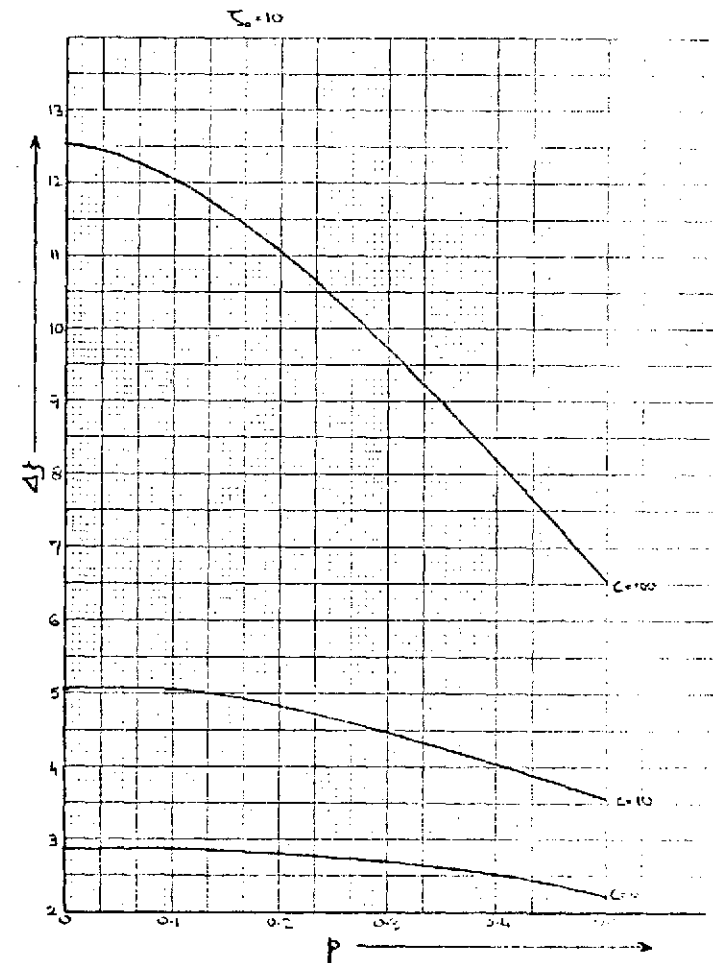


Fig. 6. Amount of slow growth  $\Delta\zeta = \zeta_f - \zeta_0$  vs. initial crack size/panel width ratio at a constant initial crack size and a variable rate sensitivity/rate of loading ratio.



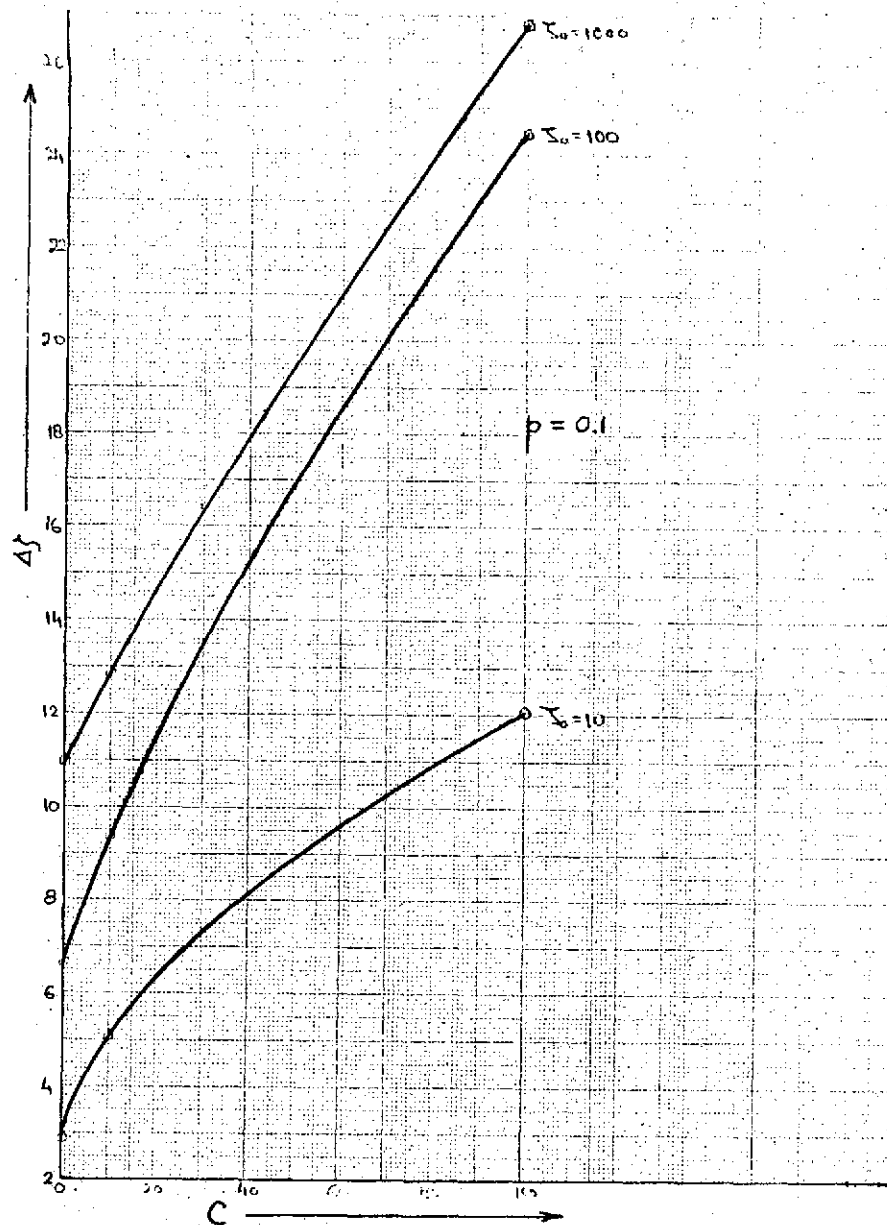


Fig. 7a

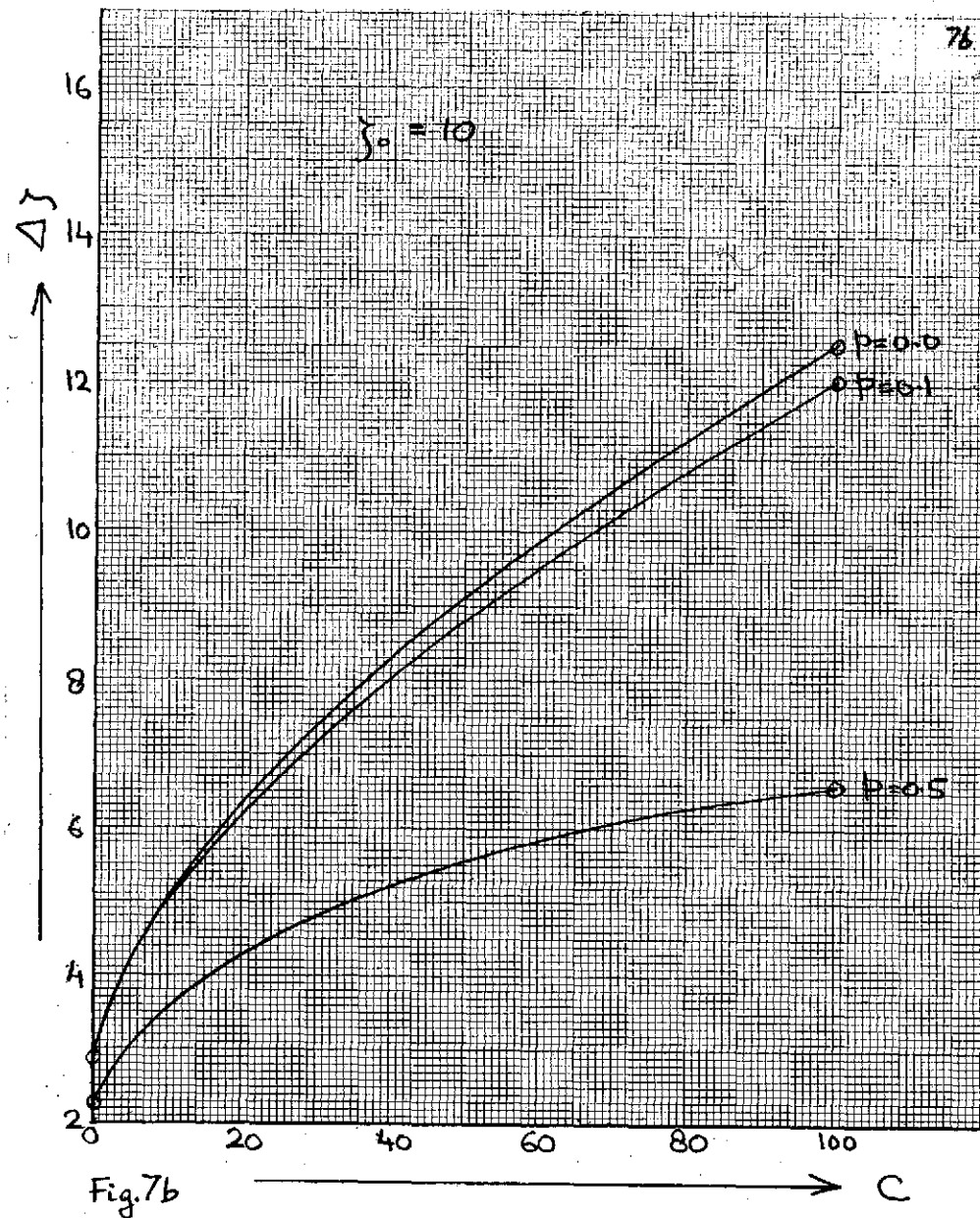
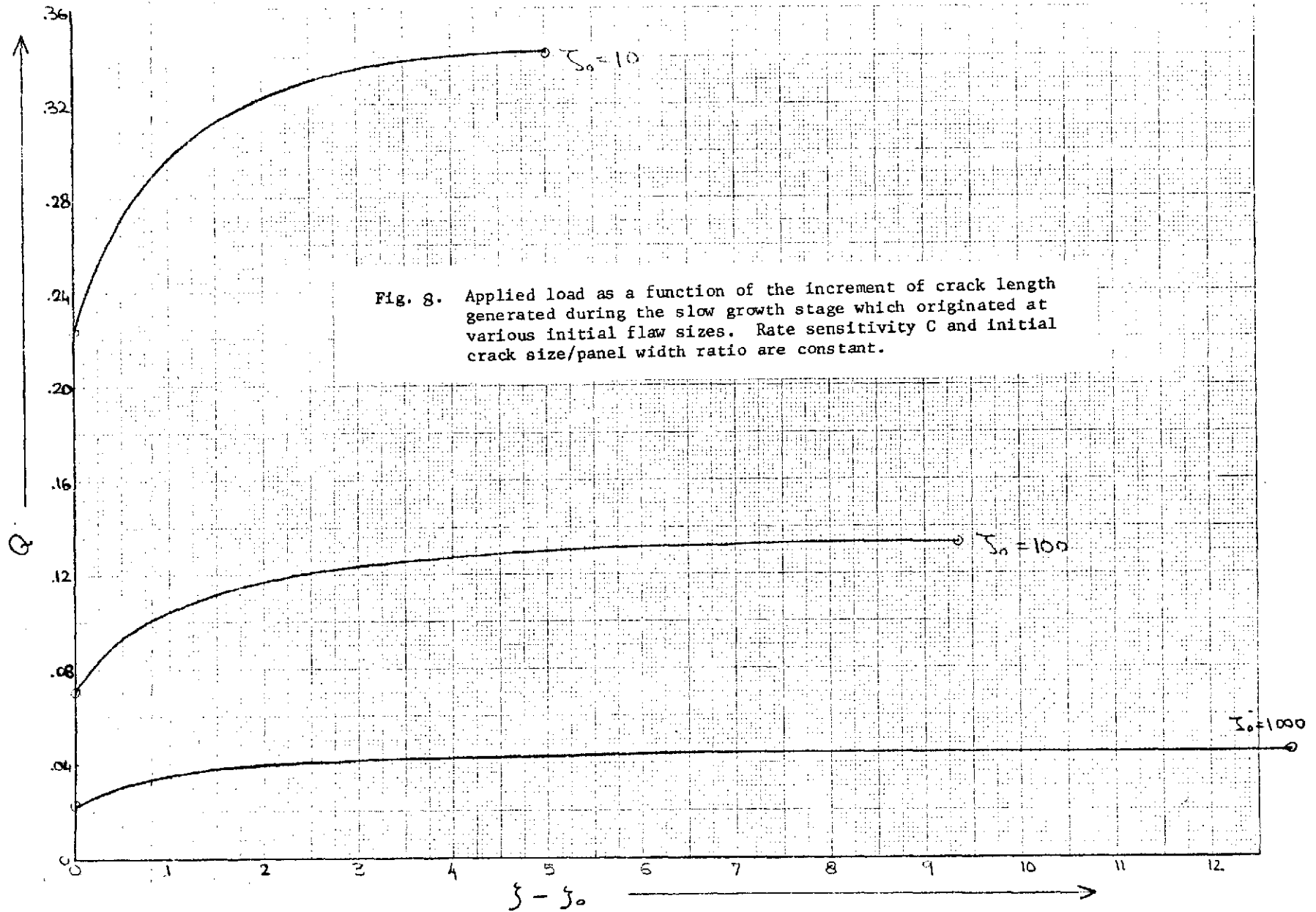


Fig. 7b

Fig. 7. Amount of slow growth  $\Delta\zeta = \zeta_f - \zeta_0$  vs. rate sensitivity/rate of loading ratio. Initial crack size is 10, 100 and 1000 in  $R_\infty$  units in Fig. 7a. In Fig. 7b the initial crack size is constant and equal  $10R_\infty$ , while the ratio  $p = (\pi/2) \times \text{crack length} / \text{panel width}$  varies.

$C=10 \quad p=0.1$



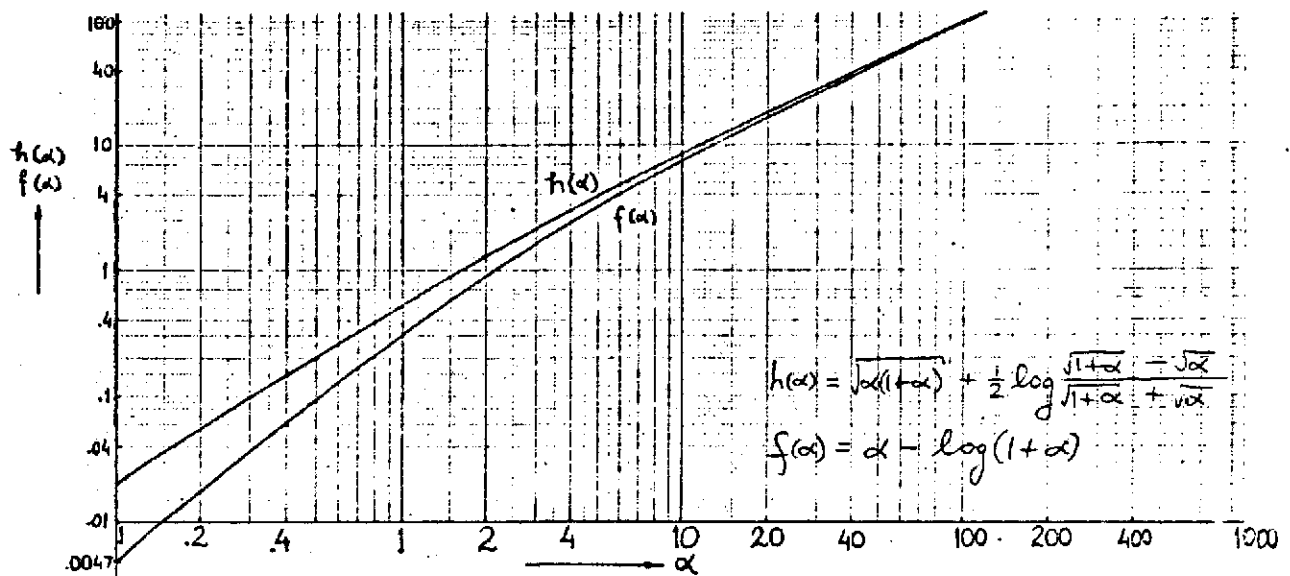


Fig.A1. Initial slope of the R-curve vs. ductility parameter  $\alpha$ ;  
 $h(\alpha)$  denotes the present solution,  $f(\alpha)$  is Rice's solution.

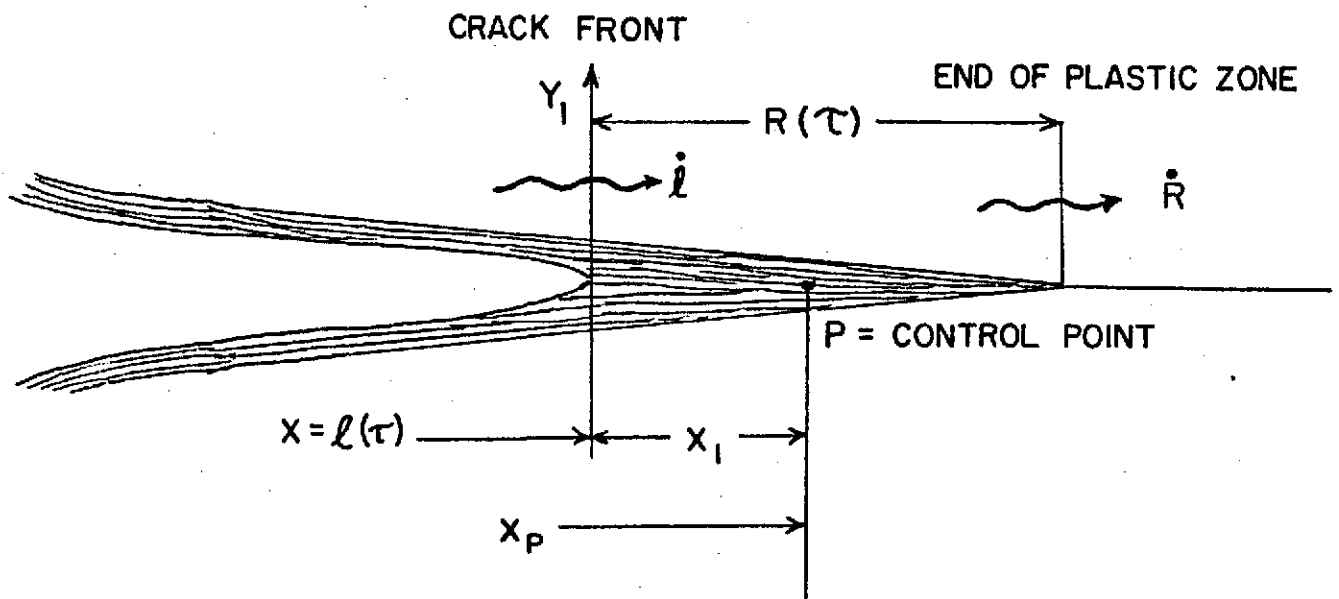
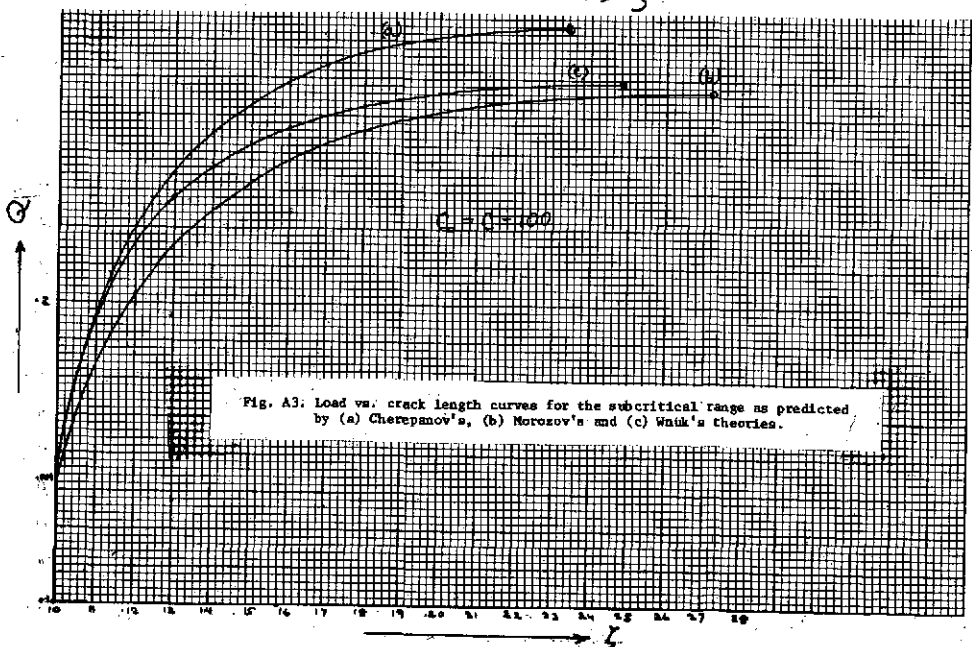
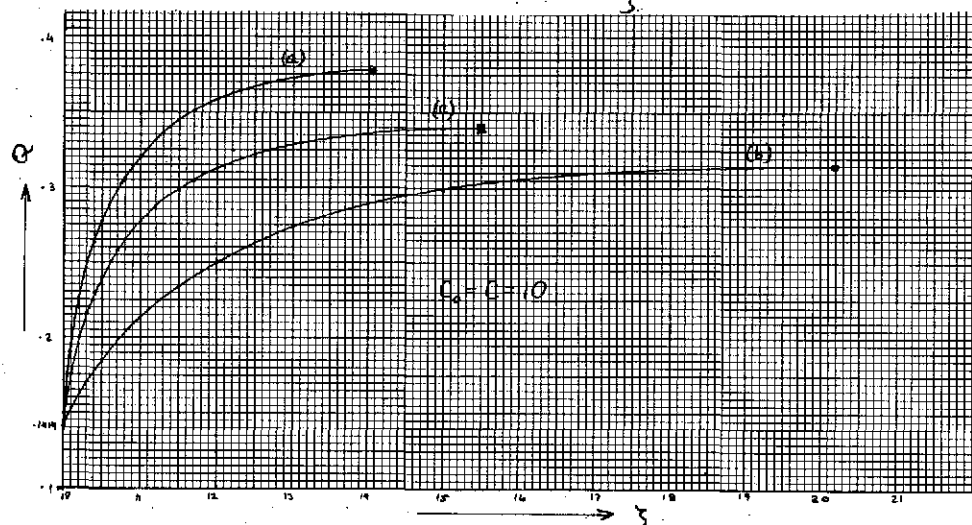
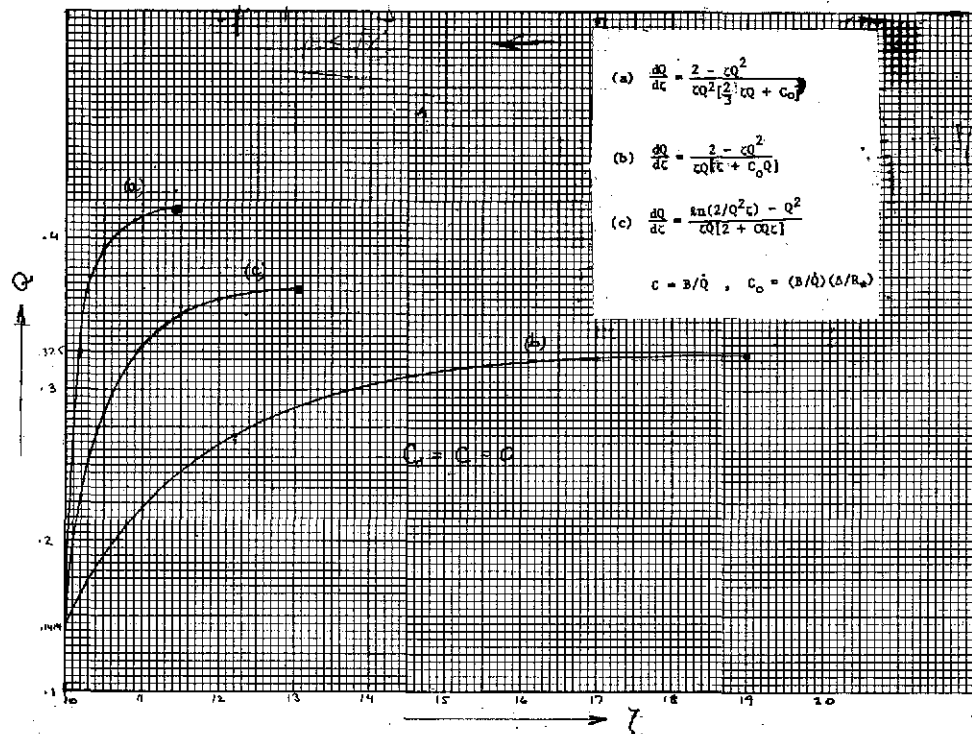


Fig. A2. Front of an advancing crack and the associated yielded (or crazed) zone. P is the control point.



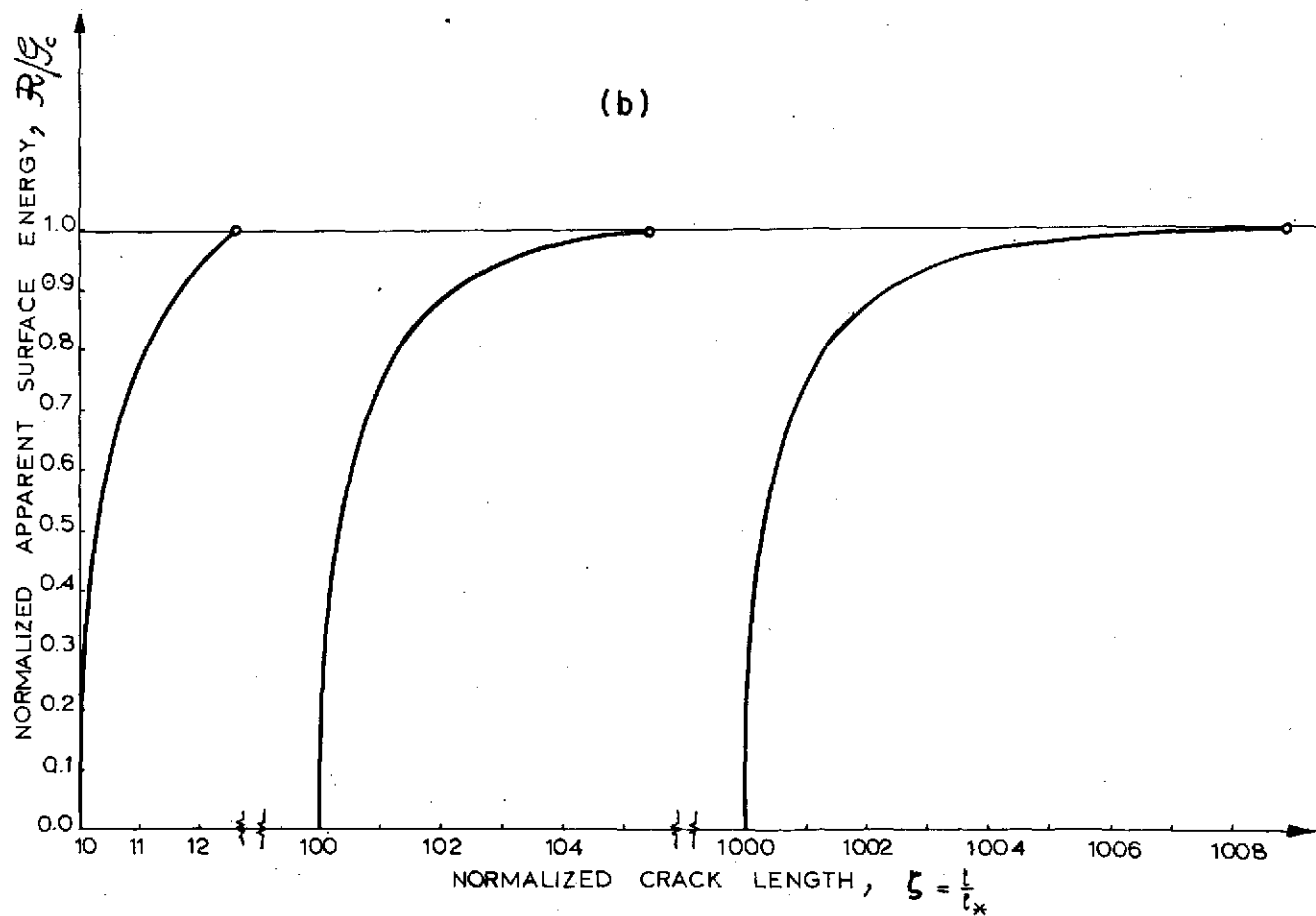
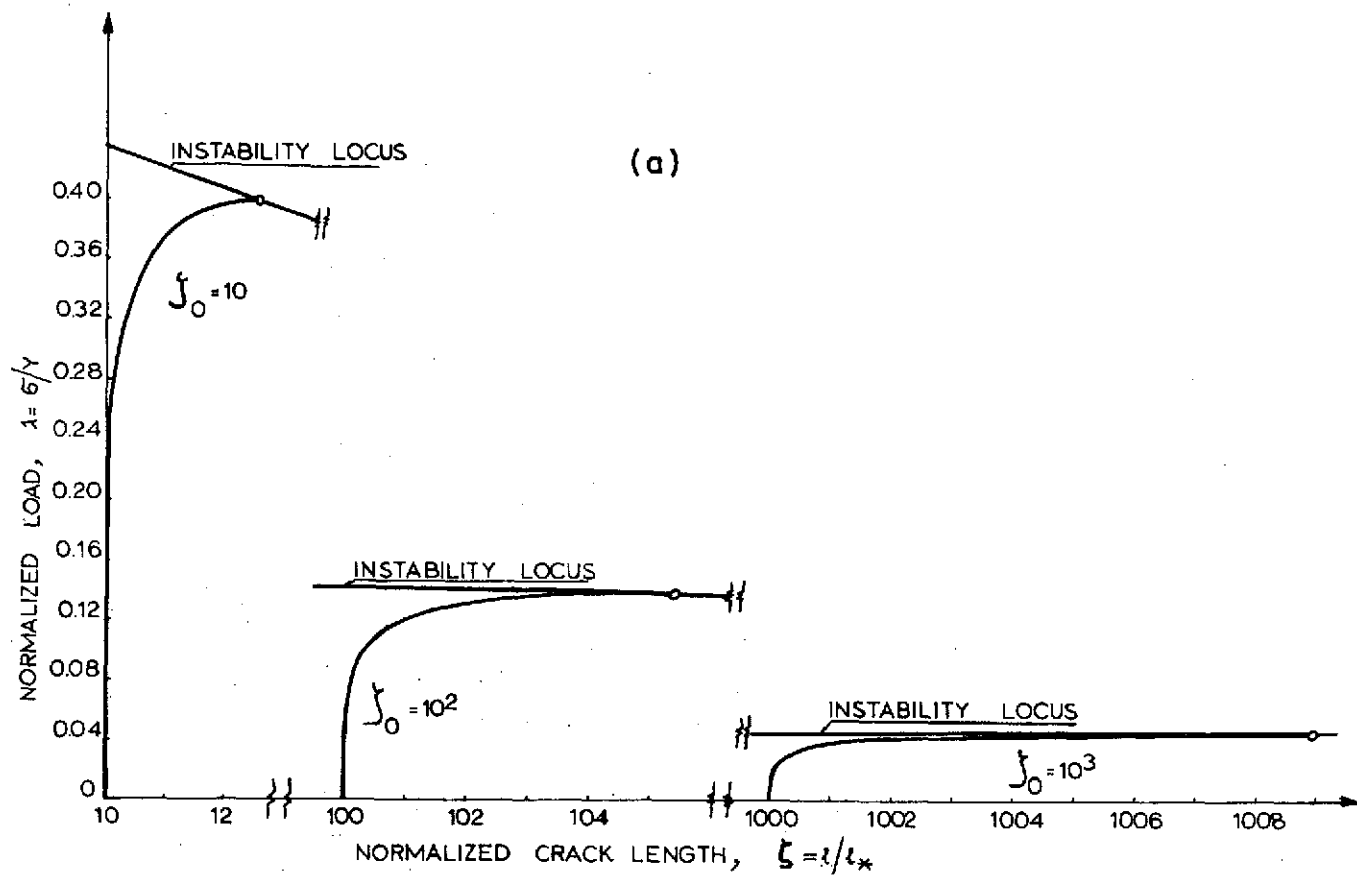


FIG.A4 SLOW GROWTH UNDER MONOTONICALLY INCREASING LOAD  
 (a) LOAD VS. CRACK LENGTH CURVES,  
 (b) APPARENT SURFACE ENERGY.

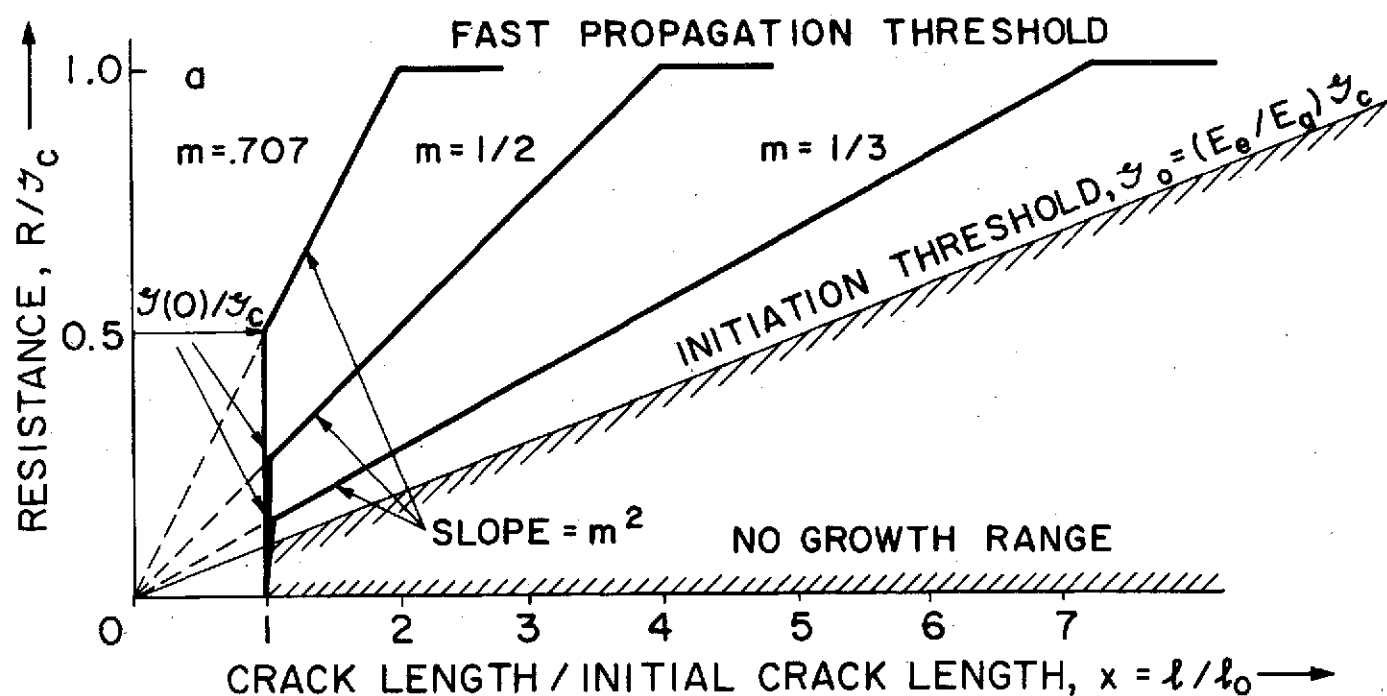
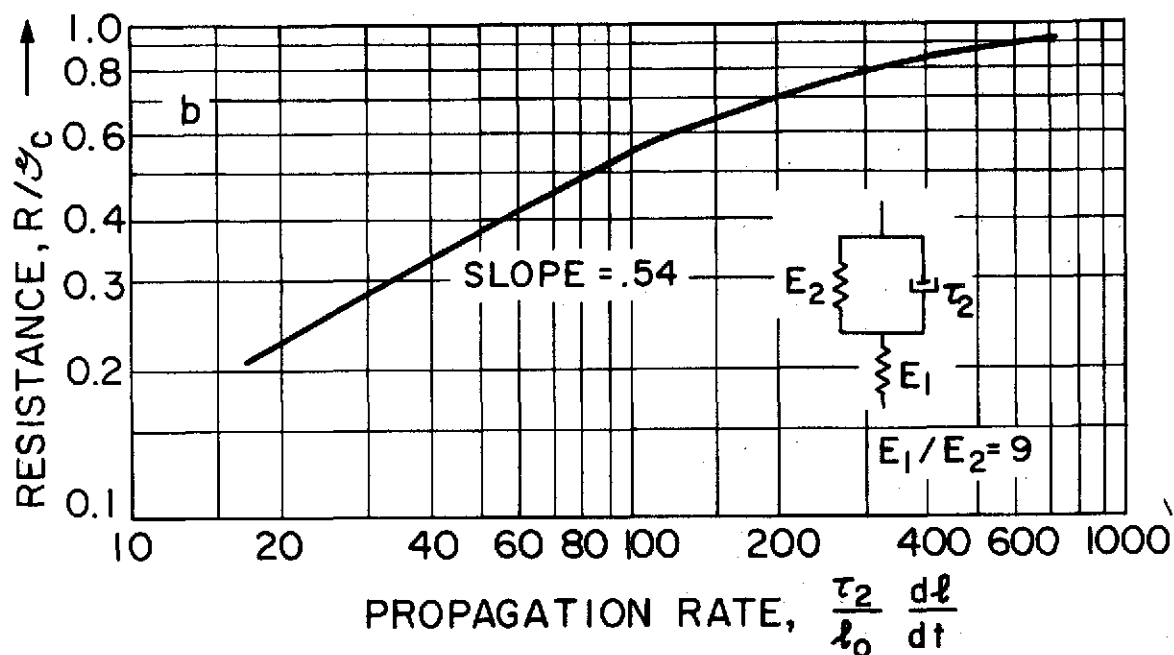


FIG. A5 APPARENT SURFACE ENERGY (RESISTANCE) OF A VISCOELASTIC SOLID IN THE SUBCRITICAL RANGE, AND UNDER SUSTAINED LOAD,  $m$  = APPLIED LOAD/ GRIFFITH LOAD